COMBINATORIAL ASPECTS OF PARTIALLY ORDERED SETS

BERTON A. EARNSHAW

Dedicated to my wife Tiraje, who I love.

ABSTRACT. This set of notes is prepared for the Meander Group (MG) at Brigham Young University. Its purpose is to introduce MG to:
(1) the basic definitions and theorems of partially ordered set theory and
(2) the various combinatorial methods associated with partially ordered sets.

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Especial thanks to R. P. Stanley, from whose book [15] most of these notes are taken.

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1. Partially Ordered Sets

We begin our study of partially ordered sets with some basic definitions, examples and results.

1.1. Partially Ordered Sets.

Definition 1.1.1. A *partially ordered set* (or *poset* for short) is an ordered pair $(P, \leq)$, denoted ambiguously by P, consisting of a set P and relation $\leq$ on P satisfying the following three properties:

(1) for all $x \in P$, $x \leq x$ (reflexivity).
(2) for all $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$ (anti-symmetry).
(3) for all $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

Remark Obviously, the notation $x \geq y$ means $y \leq x$ and the notation $x < y$ is used when both $x \leq y$ and $x \neq y$. Similarly, the notation $x > y$ means $y < x$. When it is ambiguous to which poset the relation belongs, we will write $\leq_P$ instead of $\leq$.

Example 1.1.1. The following are standard examples of posets. Given $m, n \in \mathbb{N}$, define $[m, n] := \{m+1, m+2, \ldots, n\}$. If $m = 1$, let $[n] := [1, n] := \{1, 2, \ldots, n\}$.

(1) The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, together with their linear orderings, are all posets, denoted $\underline{\mathbb{N}}$, $\underline{\mathbb{Z}}$, $\underline{\mathbb{Q}}$ and $\underline{\mathbb{R}}$, respectively.
(2) Given $n \in \mathbb{N}$, the poset $\mathbf{n}$ is the set $[n]$ ordered by magnitude; i.e., the linear ordering $1 < 2 < \ldots < n$.
(3) Given $n \in \mathbb{N}$, the poset B_n is the power set of $[n]$ ordered by inclusion; i.e., $X \leq_{B_n} Y$ if and only if $X \subseteq Y$.
(4) Given $n \in \mathbb{N}$, the poset D_n is the set of positive divisors of n ordered by divisibility; i.e., $x \leq_{D_n} y$ if and only if x divides y.
(5) Given $n \in \mathbb{N}$, the poset Π_n is the set of partitions of $[n]$ ordered by refinement; i.e., $\pi \leq_{\Pi_n} \sigma$ if and only if for each $A \in \pi$ there is $B \in \sigma$ s.t. $A \subseteq B$. π is then a *refinement* of σ.

Definition 1.1.2. A poset P is *finite* if P is finite.

Remark Notice that the posets (2) through (5) from Example 1.1.1 are finite.

1.2. **Subposets.**

Definition 1.2.1. A *weak subposet* of the poset P is a poset Q s.t. $Q \subseteq P$ and if $x \leq_Q y$, then $x \leq_P y$. If also $Q = P$, then P is a *refinement* of Q. Q is an *induced subposet* (or *subposet* for short) of P if also $\leq_Q = \leq_P|_{Q \times Q}$. If $R \subseteq P$, then $(R, \leq_P|_{R \times R})$ is the subposet *induced* by P (or the relation of P) on R.

Example 1.2.1. The following are examples of subposets of the posets defined in Example 1.1.1.

(1) Given $n \in \mathbb{N}$ and $k \in [n]$, $\mathbf{k}$ is a subposet of $\mathbf{n}$.
(2) Given $n \in \mathbb{N}$ and $k \in [n]$, B_k is a subposet of B_n.
(3) Given $n \in \mathbb{N}$ and $k \in [n]$ s.t. k divides n, D_k is a subposet of D_n.
(4) Given $k, n \in \mathbb{N}$, define the poset $\mathrm{NC}_{k,n}$ as the subposet of Π_{kn} s.t each partition $\pi \in \mathrm{NC}_{k,n}$ is non-crossing (i.e., if $B, B' \in \pi$ and $a < b < c < d$ s.t. $a, c \in B$ and $b, d \in B'$, then $B = B'$) and each block of π has cardinality divisible by k.

Remark In the case that $k = 1$ from part (4) above, define $\mathrm{NC}_n := \mathrm{NC}_{1,n}$.

Theorem 1.2.1. *If P is a finite poset, then there are exactly $2^{|P|}$ subposets of P.*

Proof. Assume the poset P is finite. Since P is finite, there are exactly $2^{|P|}$ subsets of P. Given $P' \subseteq P$, there is only one relation $\leq'$ s.t. $(P', \leq')$ is a subposet of P, namely $\leq' = \leq|_{P' \times P'}$. Therefore, there are exactly $2^{|P|}$ subposets of P. □

Definition 1.2.2. If $x \leq y$ in the poset P, then the *closed interval* (or *interval* for short) from x to y, denoted ambiguously by $[x, y]$, is the subposet induced by P on the set $[x, y] := \{z \in P \mid x \leq z \leq y\}$. The *open interval* from x to y, denoted ambiguously by (x, y), is the subposet induced by P on the set $(x, y) := \{z \in P \mid x < z < y\}$. The collection of all intervals of P is denoted $\mathrm{Int}(P)$.

Remark Notice that $[x, x] = \{x\}$ and $(x, x) = \emptyset$. If it is ambiguous as to which poset $[x, y]$ or (x, y) is a subposet, we will write $[x, y]_P$ or $(x, y)_P$ instead.

Lemma 1.2.1. *A poset P is determined by the collection $\mathrm{Int}(P)$.*

Proof. This follows from the fact that $(P, \leq_P) = \bigcup_{I \in \mathrm{Int}(P)} (I, \leq_P|_{I \times I})$. □

1.3. Locally Finite Posets.

Definition 1.3.1. A poset P is *locally finite* if every interval of P is finite.

Theorem 1.3.1. *Every finite poset is locally finite.*

Proof. Assume the poset P is finite. Suppose it is not locally finite. Then for some $I \in \mathrm{Int}(P)$, I is not finite. But I is a subposet of P, which implies the contradiction P is not finite. Therefore, P is locally finite. □

Warning! The converse of this theorem is not always true! For instance, $\underline{\mathbb{N}}$ is a locally finite, but infinite, poset.

Definition 1.3.2. If $x < y$ and $(x, y) = \emptyset$ in the poset P, then y *covers* x.

Lemma 1.3.1. *A finite poset is determined by its covering relations.*

Proof. Assume P is a finite poset. Suppose P is not determined by its covering relations. Then there exist $x, y \in P$ s.t. for all $w, z \in [x, y]$, w does not cover z. Choose $p_1 \in (x, y)$. Such an element exists since y does not cover x. Since $[x, p_1] \subseteq [x, y]$, $[x, p_1]$ is not determined by its cover relations. Now choose $p_2 \in (x, p_1)$. Continuing inductively defines an infinite subset $\{p_1, p_2, p_3, \ldots\}$ of P, implying the contradiction P is infinite. Therefore, P is determined by its covering relations. □

Warning! An infinite poset is not always determined by its covering relations! For instance, $[0, 1]$ is an interval of $\underline{\mathbb{R}}$ with no covering relations. To see this, let $x < y$ in $[0, 1]$. Since $\mathbb{R}$ is topologically connected, (x, y) is not empty. Thus y does not cover x. Since x and y were chosen arbitrarily, no element covers another in $[0, 1]$. Therefore, $[0, 1]$ is cannot be determined by its covering relations.

Theorem 1.3.2. *A locally finite poset is determined by its cover relations.*

Proof. Assume P is a locally finite poset. By Lemma 1.2.1, P is determined by $\mathrm{Int}(P)$. Given $I \in \mathrm{Int}(P)$, I is finite. Thus, by Lemma 1.3.1, I is determined by its covering relations. Therefore, P is determined by its covering relations. □

1.4. Hasse Diagrams.

Definition 1.4.1. The *Hasse Diagram* of a finite poset P is the graph whose vertex set is P and whose edge set is the covering relations in P. If x covers y in P, then x is drawn with a higher horizontal coordinate than y.

Example 1.4.1. The following figures are examples of Hasse diagrams.

FIGURE 1. Hasse diagram of **3**

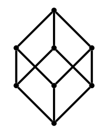

FIGURE 2. Hasse diagram of B_3

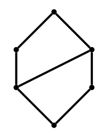

FIGURE 3. Hasse diagram of D_{12}

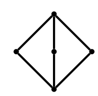

FIGURE 4. Hasse diagram of Π_3

1.5. **Minimal and Maximal Elements.**

Definition 1.5.1. An element x of a poset P is *minimal* if there is no element $y \in P$ s.t. $y < x$. Similarly, x is *maximal* if there is no element $z \in P$ s.t. $x < z$.

Lemma 1.5.1. *Let x and y be distinct minimal (maximal) elements of a poset P. Then x and y are incomparable.*

Proof. Assume x and y are minimal elements of the poset P. Suppose x and y are comparable. WLOG assume $x \leq y$. Since x is minimal, $x \not< y$, implying the contradiction $x = y$. Therefore, x and y are incomparable. □

Lemma 1.5.2. *Let P be a finite poset. Then the set of minimal (maximal) elements of P is nonempty and finite.*

Proof. Assume P is a finite poset. Let M be the set of minimal elements of P. Since $M \subseteq P$ and P is finite, M must be finite. Suppose M is empty. Given any $x_1 \in P$, x_1 is not minimal. Thus there exists $x_2 \in P$ s.t. $x_2 < x_1$. Also x_2 is not minimal, so there exists $x_3 \in P$ s.t. $x_3 < x_2 < x_1$. Continuing inductively yields an infinite subset $\{x_1, x_2, x_3, \ldots\}$ of P, contradicting the finiteness of P. Therefore, M is nonempty. □

Definition 1.5.2. If an element x of a poset P is s.t. for all $y \in P$, $x \leq y$, then x is called the *infimum* of P, and is denoted $\widehat{0}$. If the element x is s.t. for all $y \in P$, $y \leq x$, then x is called the *supremum* of P, and is denoted $\widehat{1}$. The poset $\widehat{P}$ is formed by adjoining to P an infimum and supremum (in spite of an infimum or supremum that P may already possess).

Remark If it is ambiguous as to which poset the infimum or supremum belongs, we will write $\widehat{0}_P$ and $\widehat{1}_P$, respectively. If P already possesses an infimum, then $\widehat{0}_P$ covers $\widehat{0}_{\widehat{P}}$. Similarly, if P possesses a supremum, then $\widehat{1}_{\widehat{P}}$ covers $\widehat{1}_P$.

1.6. **Chains.**

Definition 1.6.1. Two elements x and y in the poset P are *comparable* if $x \leq y$ or $y \leq x$; otherwise x and y are *incomparable*.

Definition 1.6.2. A poset P is a *chain* (or *totally ordered set* or *linearly ordered set*) if every pair of elements is comparable. A nonempty subset C of P is a *chain of* P if $(C, \leq_P|_{C\times C})$ is a chain. The collection of all chains of P is denoted $\mathrm{Chn}(P)$.

Definition 1.6.3. If a chain C of a poset P is finite, then the *length* of C is $\mathrm{len}(C) := |C| - 1$. If P is locally finite and the length of each chain is bounded by some $N \in \mathbb{N}$, then the *length* (or *rank*) of P is $\mathrm{len}(P) := \max\{\mathrm{len}(C) \mid C \in \mathrm{Chn}(P)\}$.

Definition 1.6.4. A chain C of P is *saturated* (or *unrefinable* or *connected*) if for all $x \leq y$ in C and $z \in [x, y] \setminus C$, $C \cup \{z\}$ is not a chain of P. C is *maximal* if there is no chain C' of P s.t. $C \subsetneq C'$.

Example 1.6.1. The following are examples of chains.

(1) Given $n \in \mathbb{N}$ and $k \in [n]$, $\mathbf{n}$ is a chain of length $n-1$. $[k]$ is a saturated chain of $\mathbf{n}$ of length $k-1$, and is maximal when $k = n$.
(2) Given $n \in \mathbb{N}$ and $k \in [n]$, $\mathrm{len}(B_n) = n$ and the collection $\{\emptyset, [1], [2], \ldots, [k]\}$ is a saturated chain of B_n of length k. The collection is maximal when $k = n$. B_n is a chain if and only if $n = 1$.
(3) Given $n \in \mathbb{N}$, D_n is a chain if and only if $n = p^k$ for some $p, k \in \mathbb{N}$, p prime. In this case $\mathrm{len}(D_{p^k}) = k$.
(4) Given $k, n \in \mathbb{N}$, $\mathrm{len}(\Pi_n) = \mathrm{len}(\mathrm{NC}_{k,n}) = n - 1$ and the collection
$$1/2/\ldots/n < 1,2/\ldots/n < \cdots < 1,2,\ldots,n$$
is a maximal chain of Π_n and
$$1,\ldots,k/k+1,\ldots,2k/\ldots/(n-1)k+1,\ldots,nk <$$
$$1,\ldots,2k/\ldots/(n-1)k+1,\ldots,nk < \cdots < 1,2,\ldots,nk$$
is a maximal chain of $\mathrm{NC}_{k,n}$, both of length $n-1$. Π_n and $\mathrm{NC}_{k,n}$ are chains if and only if $n \leq 2$.

Lemma 1.6.1. *Every maximal chain of a poset is saturated.*

Proof. Assume C is a chain of a poset P. Suppose C is not saturated. Then there exists $z \in P \setminus C$ s.t. $C \cup \{z\}$ is a chain. But then $C \subsetneq C \cup \{z\}$, contradicting the maximality of C. Therefore, C is saturated. □

Theorem 1.6.1. *Let $C = \{x_0, x_1, \ldots, x_n\}$ be a finite chain of a poset P of length $n \geq 1$ s.t. $x_0 < x_1 < \cdots < x_n$. C is saturated if and only if for all $i \in [n]$, x_i covers x_{i-1}.*

Proof. Assume C and P are as in the conditions of the lemma.

($\Rightarrow$) Assume C is saturated. Suppose that for some $i \in [n]$, x_i does not cover x_{i-1}. Then (x_{i-1}, x_i) is nonempty. Given $y \in (x_{i-1}, x_i)$, $C \cup \{y\}$ is a chain of P since $x_0 < \ldots < x_{i-1} < y < x_i < \ldots < x_n$. But this contradicts the saturation of C. Therefore, x_i covers x_{i-1}.

($\Leftarrow$) Assume that for all $i \in [n]$, x_i covers x_{i-1}. Then (x_{i-1}, x_i) is empty. Thus there is no $y \in [x_{i-1}, x_i] \setminus C$ s.t. $C \cup \{y\}$ is a chain. Therefore, C is saturated. □

1.7. Poset Isomorphisms and Duality.

Definition 1.7.1. A function $\phi : P \to Q$ from the poset P into the poset Q is *isotone* (or *order-preserving*) if $x \leq_P y$ implies $\phi(x) \leq_Q \phi(y)$. If ϕ is also a bijection whose inverse is isotone, then ϕ is a *poset isomorphism* from P into Q. If such a bijection exists, then the posets P and Q are said to be *isomorphic*, denoted $P \cong Q$.

Example 1.7.1. Various cases of the posets we have considered are isomorphic.

(1) Given $p, k \in \mathbb{N}$ s.t. p is prime, $\mathbf{k} \cong D_{p^{k-1}}$.
(2) Given $n \in \mathbb{N}$ s.t. n is square-free, $B_n \cong D_n$.

Definition 1.7.2. The *dual poset* (or *dual* for short) of the poset P is the poset P^* s.t. $x \leq_{P^*} y$ if and only if $y \leq_P x$. P is *self-dual* if $P \cong P^*$.

Remark Notice that all of the posets of Example 1.1.1, with the exception of D_n, are self-dual. However, D_n is self-dual if n is square-free or $n = p^k$ for some $p, k \in \mathbb{N}$, p prime (this follows from the observations made in Example 1.7.1).

1.8. Antichains and Order Ideals.

Definition 1.8.1. A set A is an *antichain* (or *Sperner family* or *clutter*) of a poset P if $A \subseteq P$ and any pair of elements of A is incomparable in P. The collection of all antichains of P is denoted Anti(P), and the poset induced by inclusion on Anti(P) is denoted ambiguously by Anti(P).

Definition 1.8.2. A set I is an *order ideal* (or *semi-ideal* or *down-set* or *decreasing subset*) of a poset P if $I \subseteq P$ and for all $x \in I$ and $y \in P$, if $y \leq x$, then $y \in I$. Similarly, the set I is a *dual order ideal* (or *filter*) if $I \subseteq P$ and for all $x \in I$ and $y \in P$, if $x \leq y$, then $y \in I$. The collection of all order ideals of P is denoted $J(P)$, and the poset induced by inclusion on $J(P)$ is denoted ambiguously by $J(P)$.

Definition 1.8.3. Let P be a poset and $A \subseteq P$. The order ideal *generated* by A in P is the set $\langle A \rangle := \{x \in P \mid x \leq y \text{ for some } y \in P\}$, and the poset induced by P on $\langle A \rangle$ is denoted $\langle A \rangle$. $\langle A \rangle$ is *finitely generated* if A is finite. If for some $x \in P$, $A = \{x\}$, then $\langle A \rangle$ is the *principal order ideal* generated by x and is denoted Λ_x. Similarly, the *principal dual order ideal* generated by x is the set $\mathrm{V}_x := \{y \in P \mid x \leq y\}$.

Lemma 1.8.1. *Let P be a finite poset, $I \in J(P)$, and I the poset induced by P on I. Then I is finitely generated by the maximal elements of I.*

Proof. Assume P is finite. Given $I \in J(P)$, let I denote the poset induced by P on I, and let G be the maximal elements of I. I must also be finite, so by Lemma 1.5.2, G is nonempty and finite. Given $x \in \langle G \rangle$, there exists $g \in G$ s.t. $x \leq g$. Since also $g \in I$, it follows that $x \in I$. Therefore, $\langle G \rangle \subseteq I$.

Now, given $i \in I$, i is either a maximal element of I of not; i.e., either $i \in G$ or there exists some $h \in G$ s.t. $i < h$. This implies $i \in \langle G \rangle$, and so $I \subseteq \langle G \rangle$. This, together with the result above, gives $I = \langle G \rangle$. Therefore, I is finitely generated by the maximal elements of I. $\square$

Theorem 1.8.1. *Let P be a finite poset. Then* Anti$(P) \cong J(P)$.

Proof. Assume P is a finite poset. Let $\phi : \text{Anti}(P) \to J(P)$ be a function defined for all $A \in \text{Anti}(P)$ by $\phi(A) = \langle A \rangle$. Clearly, ϕ is well-defined.

Given $A \in \text{Anti}(P)$, Lemma 1.5.2 and the fact that no two elements of A are comparable imply A is the set of maximal elements of $\langle A \rangle$. Thus for any other $B \in \text{Anti}(P)$ s.t. $A \neq B$, $\langle A \rangle$ and $\langle B \rangle$ have different maximal elements, implying $\langle A \rangle \neq \langle B \rangle$. Therefore, ϕ is injective.

Given $I \in J(P)$, I is finite since P is finite. Thus, by Lemma 1.8.1, I is finitely generated. The generators of I are the maximal elements of I and form, by Lemma 1.5.1, an antichain D of P s.t. $I = \langle D \rangle$. Thus ϕ is surjective. Since ϕ is also injective, it is bijective.

Given $E, F \in \text{Anti}(P)$, it is clear that $E \subseteq F$ if and only if $\langle E \rangle \subseteq \langle F \rangle$. Therefore, ϕ is an isomorphism, implying $\text{Anti}(P) \cong J(P)$. □

Warning! The assumption that P is finite is important, as it guarantees that every order ideal is finitely, and thus uniquely, generated. If P is not finite, then there may be some order ideals of P which are not finitely generated. Consider, for instance, $\underline{\mathbb{R}}$. $J(\underline{\mathbb{R}}) = \{(-\infty, x] |\ x \in \mathbb{R}\} \cup \{(-\infty, x) |\ x \in \mathbb{R}\}$. Notice that an order ideal of the form $(-\infty, x)$ cannot be finitely generated. $\text{Anti}(\underline{\mathbb{R}}) = \{x |\ x \in \mathbb{R}\}$, hence $\text{Anti}(\underline{\mathbb{R}}) \prec J(\underline{\mathbb{R}})$. Therefore, $\text{Anti}(\underline{\mathbb{R}}) \ncong J(\underline{\mathbb{R}})$.

1.9. **Operations on Posets.** Throughout this subsection we assume P and Q are posets.

Definition 1.9.1. Considering P and Q as disjoint, the *cardinal sum* (or *direct sum* or *sum*) of P and Q is the poset $P + Q := (P \cup Q, \leq_{P+Q})$ s.t. $x \leq_{P+Q} y$ if and only if $x \leq_P y$ or $x \leq_Q y$. Given $n \in \mathbb{N}$, the sum of P with itself n times is denoted nP. A poset is *connected* if it is not the sum of two nonempty posets.

Definition 1.9.2. The *cardinal product* (or *direct product* or *cartesian product* or *product*) of P and Q is the poset $P \times Q := (P \times Q, \leq_{P \times Q})$ s.t. $(x, y) \leq_{P \times Q} (x', y')$ if and only if $x \leq_P x'$ and $y \leq_Q y'$. Given $n \in \mathbb{N}$, the product of P with itself n times is denoted P^n.

2. Graded Posets

This section will introduce the concept of a graded poset and a few associated results.

2.1. **Rank Functions.**

Definition 2.1.1. A *rank function* of a poset P is function $\rho : P \to \mathbb{N} \cup \{0\}$ having the following properties:

(1) if x is minimal, then $\rho(x) = 0$.
(2) if y covers x, then $\rho(y) = \rho(x) + 1$.

Warning! Not all posets possess a rank function! For instance, the locally finite chain $\underline{\mathbb{Z}}$ does not. To see this, suppose that ρ is a rank function for $\underline{\mathbb{Z}}$. Given any $z \in \mathbb{Z}$, $\rho(z) = k$ for some $k \in \mathbb{N} \cup \{0\}$. Notice $k \neq 0$ since no element of $\underline{\mathbb{Z}}$ is minimal. Since z covers $z - 1$ covers ... covers $z - k$, $\rho(z - k) = \rho(z - k + 1) - 1 = \cdots = \rho(z) - k = k - k = 0$, implying the contradiction $z - k$ is minimal. Therefore, $\underline{\mathbb{Z}}$ does not possess a rank function.

Lemma 2.1.1. *Every finite chain possesses a unique rank function.*

Proof. Assume $C = \{x_0, x_1, \dots, x_n\}$ is a finite chain of length n s.t. $x_0 < x_1 < \cdots < x_n$. Then x_0 is a minimal element of C, and for all $i \in [n]$, x_i covers x_{i-1}. Define $\rho : C \to [0, n]$ by $\rho(x_i) = i$ and for all $i \in [n]$. Then ρ satisfies the properties of a rank function for C.

Suppose ρ' is another rank function for C different from ρ. Then for some $i \in [n]$, $\rho(x_i) \neq \rho'(x_i)$. WLOG assume $\rho(x_i) < \rho'(x_i)$. Then $\rho'(x_0) = \rho'(x_1) - 1 = \cdots = \rho'(x_i) - i > \rho(x_i) - i = i - i = 0$, contradicting the fact that $\rho'(x_0) = 0$. Therefore, ρ is unique. □

2.2. **Graded Posets.**

Definition 2.2.1. If every maximal chain of the poset P has the same length $n \in \mathbb{N} \cup \{0\}$, then P is *graded of rank n*.

Remark While the rank of a graded poset must be finite, the poset itself does not need to be so. For instance, $(\mathbb{N}, =)$ is an infinite graded poset of rank 0.

Theorem 2.2.1. *Every graded poset possesses a unique rank function.*

Proof. Assume P is a graded poset of rank n. Let $C = \{x_0, x_1, \dots, x_n\}$ be an arbitrary maximal chain of P s.t. $x_0 < x_1 < \cdots < x_n$. By lemma 2.1.1 there is a unique rank function $\rho_C : C \to [0, n]$ for C.

Let $C' = \{x'_0, x'_1, \dots, x'_n\}$ be any other maximal chain s.t. $x'_0 < x'_1 < \cdots < x'_n$ and $C \cap C'$ is nonempty. Let $\rho_{C'}$ be the unique rank function for C' and suppose that for some $x \in C \cap C'$, $\rho_C(x) \neq \rho_{C'}(x)$. Then for some $i, j \in [n] \cup \{0\}$ s.t. $i \neq j$, $x = x_i = x'_j$. WLOG assume $i < j$. This implies $x'_0 < \cdots < x'_j = x_i < \cdots < x_n$ in P. But then $\{x'_0, \dots, x'_j = x_i, \dots, x_n\}$ is a chain of P of length $j + n - i > n$, which contradicts the fact that maximal chains in P have length n. Therefore, $\rho_C(x) = \rho_{C'}(x)$, and so ρ_C and $\rho_{C'}$ agree on all of $C \cap C'$.

Since $P = \cup\{C \subseteq P \mid C \text{ is a maximal chain of } P\}$,

$$\rho := \cup\{\rho_C \mid C \text{ is a maximal chain of } P\}$$

is a rank function from P into $[0, n]$. The uniqueness of each ρ_C implies the uniqueness of ρ. □

Warning! Having finite length or possessing an infimum and supremum is not enough to guarantee a unique rank function! For instance, the poset $\widehat{\mathbf{2} + \mathbf{1}}$ has length 3 and possesses an infimum and supremum, yet no rank function can be assigned to it.

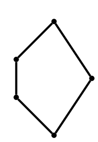

FIGURE 5. Hasse diagram of $\widehat{\mathbf{2} + \mathbf{1}}$

Definition 2.2.2. Let P be a graded poset with rank function ρ. Then for all $x \in P$, x has *rank* $\rho(x)$.

Example 2.2.1. Almost all of the posets considered so far have been graded.

(1) Given $n \in \mathbb{N}$, $\mathbf{n}$ is graded of rank $n - 1$. Given $k \in [n]$, k has rank $k - 1$.

(2) Given $n \in \mathbb{N}$, B_n is graded of rank n. Given $A \in B_n$, A has rank $|A|$.
(3) Given $n \in \mathbb{N}$, D_n is graded of rank $d(n)$, where $d(n)$ is the number of prime divisors (counting multiplicities) of n. Given k a divisor of n, k has rank $d(k)$.
(4) Given $k, n \in \mathbb{N}$, Π_n and $\mathrm{NC}_{k,n}$ are both graded of rank $n-1$. Given π in either poset, π has rank $n - |\pi|$.

Theorem 2.2.2. *If $x \leq y$ in a graded poset P with rank function ρ, then* $\mathrm{len}([x,y]) = \rho(y) - \rho(x)$.

Proof. Assume P is a graded poset of rank n with rank function ρ. Given $x \leq y$ in P, let $C = \{x_0, x_1, \ldots, x_n\}$ be a maximal chain of P containing x and y s.t. $x_0 < x_1 < \cdots < x_n$. Then for some $i, j \in [n]$ s.t. $i < j$, $x_i = x$ and $x_j = y$. This forces $\mathrm{len}([x,y]) = j - i$, else $\mathrm{len}(C) \neq n$. By theorem 2.2.1, $\rho(x) = i$ and $\rho(y) = j$. Therefore, $\mathrm{len}([x,y]) = j - i = \rho(y) - \rho(x)$ □

2.3. **Rank-generating Function.**

Definition 2.3.1. If P is a graded poset of rank n s.t. for each $i \in [n]$, p_i is the number of elements of P of rank i, then the *rank-generating function* of P is the function $\mathrm{F}(P,x) := \sum_{i=0}^{n} p_i x^i$.

Example 2.3.1. Almost all of the posets considered so far have been graded.

(1) Given $n \in \mathbb{N}$, the rank-generating function of $\mathbf{n}$ is $\mathrm{F}(\mathbf{n}, x) = \sum_{i=0}^{n-1} x^i = 1 + x + \cdots + x^{n-1}$.
(2) Given $n \in \mathbb{N}$, the rank-generating function of B_n is $\mathrm{F}(B_n, x) = \sum_{i=0}^{n} \binom{n}{i} x^i$.
(3) Given $n \in \mathbb{N}$ square-free, $\mathrm{F}(D_n, x) = \mathrm{F}(B_n, x)$.
(4) Given $n \in \mathbb{N}$, the rank-generating function for Π_n is

$$\mathrm{F}(\Pi_n, x) = \sum_{i=0}^{n-1} S(n, n-i) x^i,$$

where $S(n,k) = \frac{1}{k!}\sum_{i=0}^{k} (-1)^{k-i}\binom{k}{i} i^n$ is a *Stirling number of the second kind* (ref. Section 4.1). Given $k \in \mathbb{N}$, the rank-generating function for $\mathrm{NC}_k n$ is

$$\mathrm{F}(\mathrm{NC}_{k,n}, x) = \sum_{i=0}^{n-1} \frac{1}{n}\binom{n}{n-i}\binom{kn}{n-i-1} x^i$$

(ref. Section 4.2).

Lemma 2.3.1. *If both P and Q have finite lengths, then* $\mathrm{len}(P \times Q) = \mathrm{len}(P) + \mathrm{len}(Q)$.

Proof. Assume P has length m and Q has length n. Given an arbitrary chain $C = \{(x_0, y_0), (x_1, y_1), \ldots, (x_l, y_l)\}$ of $P \times Q$ s.t. $(x_0, y_0) <_{P\times Q} (x_1, y_1) <_{P\times Q} \cdots <_{P\times Q} (x_l, y_l)$, it follows that $X = \{x_0, x_1, \ldots, x_l\}$ is a chain of P and $Y = \{y_0, y_1, \ldots, y_l\}$ is a chain of Q. Notice that for each $i \in [l]$, $(x_{i-1}, y_{i-1}) <_{P\times Q} (x_i, y_i)$ implies $x_{i-1} <_P x_i$ or $y_{i-1} <_Q y_i$. Since $\mathrm{len}(X) \leq m$ and $\mathrm{len}(Y) \leq n$, $x_{i-1} <_P x_i$ is true for at most m of the i's in $[l]$ and $y_{i-1} <_Q y_i$ is true for at most n of them. Therefore, $\mathrm{len}(C) = l \leq m + n$, and so every chain in the interval has length less than or equal to $m + n$.

Let $\{a_0, a_1, \ldots, a_m\}$ and $\{b_0, b_1, \ldots, b_n\}$ be chains of P and Q, respectively, of lengths m and n, respectively, s.t. $a_0 <_P a_1 <_P \cdots <_P a_m$ and $b_0 <_P b_1 <_P \cdots <_P b_n$. Then $\{(a_0, b_0), (a_1, b_0), \ldots, (a_m, b_0), (a_m, b_1), \ldots, (a_m, b_n)\}$ is a chain of $P \times Q$ of length $m + n$. This result, together with the one from the previous paragraph, implies $\mathrm{len}(P \times Q) = m + n$. □

Theorem 2.3.1. *If P is graded of rank m and Q is graded of rank n, then $P \times Q$ is graded of rank $m + n$.*

Proof. Assume P and Q are graded of rank m and n, respectively. By Lemma 2.3.1, $P \times Q$ has rank $m + n$. Let $C = \{(x_0, y_0), (x_1, y_1), \ldots, (x_l, y_l)\}$ be an arbitrary maximal chain of $P \times Q$ s.t $(x_0, y_0) <_{P \times Q} (x_1, y_1) <_{P \times Q} \cdots <_{P \times Q} (x_l, y_l)$. If $l < m + n$, then the proof of Lemma 2.3.1 asserts that for some $i \in [l]$, $x_{i-1} <_P x_i$ and $y_{i-1} <_Q y_i$. But this implies that $C \cup \{(x_{i-1}, y_i)\}$ is a chain of $P \times Q$, contradicting the maximality of C. Thus $\mathrm{len}(C) = l = m + n$. Since C was given arbitrarily, $P \times Q$ is graded of rank $m + n$. □

Corollary 2.3.1. *If both P and Q are graded, then* $\mathrm{F}(P \times Q, x) = \mathrm{F}(P, x)\mathrm{F}(Q, x)$.

Proof. Assume P and Q are graded of rank m and n, respectively, with rank-generating functions $\mathrm{F}(P, x) = \sum_{i=0}^{m} p_i x^i$ and $\mathrm{F}(Q, x) = \sum_{i=0}^{n} q_i x^i$, respectively.

Let $x \in P$ have rank k and $y \in Q$ have rank l. Let $X = \{x_0, x_1, \ldots, x_m\}$ and $Y = \{y_0, y_1, \ldots, y_n\}$ be maximal chains of P and Q, respectively, containing x and y, respectively, s.t. $x_0 <_P x_1 <_P \cdots <_P x_m$ and $y_0 <_Q y_1 <_Q \cdots <_Q y_n$. By Theorem 2.2.1, $x = x_k$ and $y = y_l$. The chain

$$C = \{(x_0, y_0), \ldots, (x_k, y_0), \ldots, (x_k, y_l), \ldots, (x_k, y_n), \ldots, (x_m, y_n)\}$$

of $P \times Q$ s.t.

$$(x_0, y_0) < \ldots < (x_k, y_0) < \ldots < (x_k, y_l) < \ldots < (x_k, y_n) < \ldots < (x_m, y_n)$$

in $P \times Q$ has length $m + n$, and so is maximal. It follows, again by Theorem 2.2.1, that (x_k, y_l) has rank $k + l$. Thus the number of elements of $P \times Q$ of rank j is $\sum_{i=0}^{j} p_i q_{j-i}$, which is the coefficient of x^j in $\mathrm{F}(P, x)\mathrm{F}(Q, x)$. Therefore, the rank-generating function for $P \times Q$ is $\mathrm{F}(P \times Q, x) = \mathrm{F}(P, x)\mathrm{F}(Q, x)$. □

3. Lattices

This section will introduce the concept of a lattice and a few associated results, including some counting results.

3.1. **Lattices.** We need a few definitions before we can define a lattice.

Definition 3.1.1. Let x and y be elements of a poset P. An element $z \in P$ is an *upper bound* of x and y if $x \leq z$ and $y \leq z$. Let $Upp(x, y)$ be the subposet of all upper bounds of x and y. If $Upp(x, y)$ possesses a minimum z, then z is the *join* (or *least upper bound*) of x and y, denoted $x \smile y$, and read "x join y."

Dually, z is a *lower bound* for x and y if $z \leq x$ and $z \leq y$. Let $Low(x, y)$ be the subposet of all lower bounds of x and y. It $Low(x, y)$ possesses a maximum z, then z is the *meet* (or *greatest lower bound*) of x and y, denoted $x \frown y$, and read "x meet y."

Remark As always, we write $\smile_P$ and $\frown_P$ instead of $\smile$ and $\frown$ when it is ambiguous as to which poset the operation belongs.

Definition 3.1.2. A *join-semilattice* is a poset s.t every pair of elements posseses a join. A *meet-semilattice* is a poset s.t. every pair of elements possesses a meet. A *lattice* is both a join- and meet-semilattice.

Definition 3.1.3. Let L be a lattice. A *sublattice* of L is a subset $M \subseteq L$ s.t. for all $x, y \in M$, $x \smile y \in M$ and $x \frown y \in M$.

Example 3.1.1. Most of the posets we have considered so far are lattices. For instance, given $k, n \in \mathbb{N}$, $\mathbf{n}$, B_n, D_n, Π_n and $\mathrm{NC}_{k,n}$ are all lattices.

Theorem 3.1.1. *Let L be a lattice. Then*

1. $\smile$ *and* $\frown$ *are both associative, commutative, and idempotent.*
2. *for all* $x, y \in L$, $x \frown (x \smile y) = x = x \smile (x \frown y)$.
3. *for all* $x, y \in L$, $x \frown y = x \Leftrightarrow x \smile y = y \Leftrightarrow x \leq y$.

Proof. Assume L is a lattice and $x, y, z \in L$. It is helpful to notice that $x \leq y$ if and only if $\min Upp(x, y) = x$ and $\max Low(x, y) = y$.

(1) $x \smile (y \smile z) \geq x$ and $x \smile (y \smile z) \geq y \smile z$. Since $y \smile z \geq y$, $x \smile (y \smile z) \geq y$. Thus $x \smile (y \smile z) \geq x \smile y$. Also, $y \smile z \geq z$, so $x \smile (y \smile z) \geq z$. Therefore, $x \smile (y \smile z) \geq (x \smile y) \smile z$. A similar argument shows that $(x \smile y) \smile z \geq x \smile (y \smile z)$. Therefore, $x \smile (y \smile z) = (x \smile y) \smile z$, proving $\smile$ is associative in L.

Since $\min Upp(x, y) = \min Upp(y, x)$, $\smile$ is commutative. Since $\min Upp(x, x) = x$, $\smile$ is idempotent. Similar arguments prove $\frown$ is associative in L, commutative and idempotent.

(2) $x \smile y \geq x$, so that $x \frown (x \smile y) = x$. Since $x \frown y \leq x$, it follows that $x \smile (x \smile y) = x$.

(3) If $x \smile y = y$, then $x \leq y$. If $x \leq y$, then $x \frown y = x$. If $x \frown y = x$, then $x \leq y$. If $x \leq y$, then $x \frown y = x$. □

Warning! In general, joins and meets do not associate with each other! Consider the following lattice:

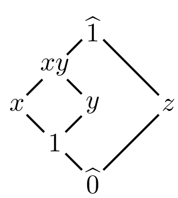

FIGURE 6. Hasse diagram of $\widehat{D_2 + \mathbf{1}}$

Here, $x \frown (y \smile z) = x$ while $(x \frown y) \smile z = \hat{1}$.

Lemma 3.1.1. *Every nonempty finite join-semilattice (meet-semilattice) possesses a supremum (infimum).*

Proof. We proceed by induction on the number of elements $n \in \mathbb{N}$ of the join-semilattice P.

$(n = 1)$ Let $P = \{x\}$. Then $\hat{1}_P = x$.

$(n = k)$ Suppose that, for some $k \in \mathbb{N} \setminus \{1\}$, the statement of the lemma is true for every join-semilattice of k elements. Let $P = \{x_1, x_2, \ldots, x_{k+1}\}$. The induction hypothesis implies $\{x_1, x_2, \ldots, x_k\}$ possesses a supremum $\hat{1}$. Then $\hat{1}_P = \hat{1} \smile x_{k+1}$.

Thus the lemma is true for join-semilattices with $k+1$ elements. Therefore, by mathematical induction, the lemma is true for every finite join-semilattice. □

Theorem 3.1.2. *If P is a finite join-semilattice (meet-semilattice) possessing an infimum (supremum), then P is a lattice.*

Proof. Assume P is a finite join-semilattice possessing an infimum. Given $x, y \in P$, $Low(x,y)$ is nonempty since it contains the infimum. Thus the subposet induced by P on $Low(x,y)$ is nonempty and finite. It is also a join-semilattice, since for all $w, z \in Low(x,y)$, $w \smile z$ exists and $w \smile z \leq x$ and $w \smile z \leq y$. Hence this induced subposet possesses a supremum z by Lemma 3.1.1. It follows then that $x \frown y = z$. Thus P is also a meet-semilattice, and therefore a lattice. □

Lemma 3.1.2. *Let L and M be lattices. Then L^*, $L \times M$ and $\widehat{L+M}$ are all lattices.*

Proof. Assume L and M are lattices. For all $x, y \in L$, it follows by definition that $x \smile_L y = x \frown_{L^*} y$ and $x \frown_L y = x \smile_{L^*} y$. Thus L^* is a lattice.

For all $(x,y),(x',y') \in L \times M$, $(x,y) \smile_{L\times M} (x',y') = (x \smile_L x', y \smile_M y')$ and $(x,y) \frown_{L\times M} (x',y') = (x \frown_L x', y \frown_M y')$. Thus $L \times M$ is a lattice.

$L+M$ is never a lattice if both L and M are nonempty, since for all $l \in L$ and $m \in M$ both $Upp(l,m)$ and $Low(l,m)$ are empty in $L+M$. In $\widehat{L+M}$, however, $Upp(l,m) = \{\widehat{1}\}$ and $Low(l,m) = \{\widehat{0}\}$. It follows from this and the fact that L and M are already lattices that $\widehat{L+M}$ is a lattice. □

3.2. **Modular Lattices.**

Theorem 3.2.1. *Let L be a finite lattice. The following conditions are equivalent:*

(1) *P is a graded and for all $x, y \in P$ its rank function ρ satisfies $\rho(x)+\rho(y) \geq \rho(x \frown y) + \rho(x \smile y)$.*
(2) *for all $x, y \in P$, if x and y both cover $x \frown y$, then $x \smile y$ covers both x and y.*

Proof. Assume L is a finite lattice.

($1 \Rightarrow 2$) Assume condition (1). Let x and y be arbitrary elements of L both covering $x \frown y$. Then $\rho(x) = \rho(x \frown y) + 1 = \rho(y)$. Suppose $x \smile y$ does not cover x. Then $\rho(x \smile y) > \rho(x) + 1$. Thus $\rho(x \frown y) + \rho(x \smile y) > \rho(y) - 1 + \rho(x) + 1 = \rho(y) + \rho(x)$, contradicting condition (1). The same contradiction arises if $x \smile y$ does not cover y. Therefore, $x \smile y$ covers both x and y.

($2 \Rightarrow 1$) Assume condition (2). Since L is finite, it has finite length $n \in \mathbb{N}$. Suppose L is not graded. Then there exist $x, y \in L$ s.t $[x,y]$ is an ungraded interval of L of minimal length $l \leq n$.

Given any element $z \in [x,y]$ covering x, $[z,y]$ is graded since it has length $l-1 < l$. Thus there must exist at least one other element of $[x,y]$ covering x. If not, then every maximal chain of $[x,y]$ is of the form $C \cup \{x\}$, where C is a maximal chain of $[z,y]$. This implies every maximal chain of $[x,y]$ has length l; i.e., the contradiction that $[x,y]$ is graded. Also, of these other covering elements, at least one of them must form an interval with y of length different than that of $[z,y]$. If not, then again all the maximal chains of $[x,y]$ are of the same length.

Therefore, let $a, b \in [x,y]$ both cover x s.t. $\text{len}([a,y]) \neq \text{len}([b,y])$. WLOG assume $\text{len}([a,y]) < \text{len}([b,y])$. It follows by (2) that $a \smile b$ covers both a and b.

Let $B = \{b, a \smile b, \ldots, y\}$ be a maximal chain of $[b, y]$ (i.e., $\text{len}(B) = \text{len}([b, y])$) containing $a \smile b$. But then $\{a, a \smile b, \ldots, y\}$ is a chain of $[a, y]$ of length $\text{len}(B)$, contradicting the fact that $\text{len}([a, y]) < \text{len}([b, y])$. Thus $[x, y]$, and hence L, must be graded. Let ρ be the rank function of L.

Now suppose condition (1)'s statement about ρ is not true. Let $L' \subseteq L$ be s.t. for all $w, z \in L'$, $\rho(w) + \rho(z) < \rho(w \frown z) + \rho(w \smile z)$. Let $L'' \subseteq L'$ be s.t. for all $w, z \in L''$, $\text{len}([w \frown z, w \smile z])$ is minimal. From L'' choose x and y s.t. $\rho(x)+\rho(y)$ is minimal. If both x and y cover $x \frown y$, then condition (2) implies $x \smile y$ covers both x and y. In this case $\rho(x) = \rho(y) = \rho(x \frown y) + 1 = \rho(x \smile y) - 1$, so that $\rho(x) + \rho(y) = \rho(x \frown y) + 1 + \rho(x \smile y) - 1 = \rho(x \frown y) + \rho(x \smile y)$. Thus x and y cannot both cover $x \frown y$.

WLOG assume x does not cover $x \frown y$. It follows that there exists $x' \in (x \frown y, x)$. Our minimality assumptions imply $\rho(x')+\rho(y) \geq \rho(x' \frown y)+\rho(x' \smile y)$. Since $x < x'$, $x' \frown y = x \frown y$. So the above inequality becomes $\rho(x') + \rho(y) \geq \rho(x \frown y) + \rho(x' \smile y)$; i.e.,

$$\rho(y) - \rho(x \frown y) \geq \rho(x' \smile y) - \rho(x').$$

Our assumptions also imply $\rho(x) + \rho(y) < \rho(x \frown y) + \rho(x \smile y)$; i.e.,

$$\rho(y) - \rho(x \frown y) < \rho(x \smile y) - \rho(x).$$

The above inequalities imply

$$\rho(x) + \rho(x' \smile y) < \rho(x') + \rho(x \smile y).$$

Let $z = x' \smile y$. Since $x' \leq x$ and $x' \leq z$, it follows that $x' \leq x \frown z$, and hence $\rho(x') \leq \rho(x \frown z)$. $x \smile (x' \smile y) = (x \smile x') \smile y = x \smile y$, so $\rho(x \smile z) = \rho(x \smile y)$. By choice $\rho(x \frown y) < \rho(x')$. The above inequality now says

$$\rho(x) + \rho(z) < \rho(x \frown z) + \rho(x \smile z)$$

with $\text{len}([x \frown z, x \smile z]) \leq \text{len}([x', x \smile y]) < \text{len}([x \frown y, x \smile y])$, contradicting the minimality assumptions for x and y. Therefore condition (1)'s statement about ρ is true [15, 103-104]. □

Definition 3.2.1. A finite lattice L is *upper semimodular* if it satisfies either one of the conditions of Theorem 3.2.1. If L^* is upper semimodular, then L is *lower semimodular*. L is *modular* if it is both upper and lower semimodular.

Example 3.2.1. Some of the lattices considered so far are modular. For instance, given $k, n \in \mathbb{N}$, $\mathbf{n}$, B_n and D_n are modular. However, Π_n and $\text{NC}_{k,n}$ are not modular for $n > 2$.

Remark The definition of a modular lattice implies the following equivalent definition: A finite lattice L is modular if and only if either of the following conditions is true:

(1) L is graded with rank function ρ s.t for all $x, y \in L$, $\rho(x) + \rho(y) = \rho(x \frown y) + \rho(x \smile y)$.
(2) for all $x, y \in L$, x and y both cover $x \frown y$ if and only if $x \smile y$ covers both x and y.

Lemma 3.2.1. *The poset* $\widehat{\mathbf{2}+\mathbf{1}}$ *is a nonmodular lattice.*

Proof. Consider again the Hasse diagram of $\widehat{\mathbf{2}+\mathbf{1}}$ (ref. Figure 5). $\widehat{\mathbf{2}+\mathbf{1}}$ is a lattice by Lemma 3.1.2. It is not modular, however, since it is not graded. □

Theorem 3.2.2. *Let L be a finite lattice. L is modular if and only if for all $x, y, z \in L$ s.t. $x \leq z$, $x \smile (y \frown z) = (x \smile y) \frown z$.*

Proof. Assume L is a finite lattice.

($\Rightarrow$) Assume L is modular and suppose that for some $x, y, z \in L$ with $x \leq z$, $x \smile (y \frown z) \neq (x \smile y) \frown z$. Clearly $y \notin [x, z]$ else $x \smile (y \frown z) = x \smile y = y = y \frown z = (x \smile y) \frown z$. Therefore, the elements y, $x \smile y$, $y \frown z$, $x \smile (y \frown z)$ and $(x \smile y) \frown z$ are distinct and form a sublattice of L isomorphic to $\widehat{\mathbf{2}+\mathbf{1}}$ (ref. Figure 7).

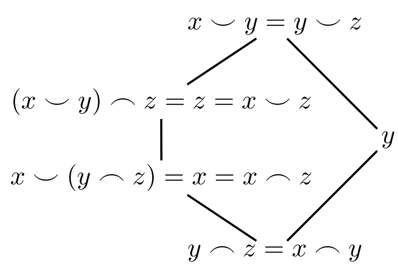

FIGURE 7. Hasse diagram of a sublattice isomorphic to $\widehat{\mathbf{2}+\mathbf{1}}$

By Lemma 3.2.1, $\widehat{\mathbf{2}+\mathbf{1}}$ is not modular. This implies the contradiction that L itself is not modular. Therefore, for all $x, y, z \in L$ s.t. $x \leq z$, $x \smile (y \frown z) = (x \smile y) \frown z$.

($\Leftarrow$) Assume that for all $x, y, z \in L$ s.t. $x \leq z$, $x \smile (y \frown z) = (x \smile y) \frown z$. Choose $x, y \in L$ s.t. x and y both cover $x \frown y$. This implies that x and y are incomparable, else both cannot cover $x \frown y$. Thus any chain of $(x \frown y, x \smile y)$ containing x is disjoint from any other chain of $(x \frown y, x \smile y)$ containing y.

Suppose $x \smile y$ does not cover x. Then there exists $z \in (x, x \smile y)$. Thus $x \frown y < x < z < x \smile y$ and $x \frown y < y < x \smile y$. It follows from the above comment about chains in $(x \frown y, x \smile y)$ that $y \frown z = x \frown y$ and $y \smile z = x \smile y$. By this and Theorem 3.1.1, $x \smile (y \frown z) = x \smile (x \frown y) = x < z = (y \smile z) \frown z = (x \smile y) \frown z$, which contradicts our assumption since $x \leq z$. Thus $x \smile y$ covers x. Similarly, $x \smile y$ covers y. The dual of the preceeding argument implies that if $x \smile y$ covers both x and y, then x and y both cover $x \frown y$. Therefore, L is modular [3, 66]. □

Remark Notice that by the preceeding theorem we can extend the concept of modularity to infinite lattices as well.

Corollary 3.2.1. *A lattice is nonmodular if and only if it contains $\widehat{\mathbf{2}+\mathbf{1}}$ as a sublattice.*

Proof. Assume L is a lattice.

($\Rightarrow$) The proof of Theorem 3.2.2 implies that a nonmodular lattice contains $\widehat{\mathbf{2}+\mathbf{1}}$ as a sublattice.

($\Leftarrow$) By Lemma 3.2.1, $\widehat{\mathbf{2}+\mathbf{1}}$ is a nonmodular lattice. Thus if L contains it as a sublattice, L cannot be modular. □

3.3. **Complemented and Atomic Lattices.**

Definition 3.3.1. Let L be a lattice with infimum and supremum. L is *complemented* if for all $x \in L$ there exists $y \in L$ s.t. $x \frown y = \widehat{0}$ and $x \smile y = \widehat{1}$. The element y is called a *complement* of x. If for all $x \in L$ the complement of x is unique, then L is *uniquely complemented*. If every interval of L is itself complemented, then L is *relatively complemented*.

Definition 3.3.2. Let L be a nonempty finite lattice. An *atom* of L is an element of L covering $\widehat{0}$, and the set of atoms of L is denoted $\mathrm{A}(L)$. L is *atomic* (or a *point lattice*) if for every $x \in L$ there exists a subset $A \subseteq \mathrm{A}(L)$ s.t. x is equal to the join of A; i.e., if $A = \{a_1, a_2, \ldots, a_n\}$, then $x = a_1 \smile a_2 \smile \cdots \smile a_n$.

Dually, a *coatom* is an element of L covered by $\widehat{1}$ and L is *coatomic* if every element is the meet of coatoms.

Remark Notice that $\widehat{1}$ is the join of $\mathrm{A}(L)$ and, by convention, $\widehat{0}$ is the join of $\emptyset$.

Lemma 3.3.1. *Let L be an atomic lattice and $a_1, a_2, \ldots, a_{n+1} \in \mathrm{A}(L)$ be a finite sequence of atoms. If $a_1 \smile a_2 \smile \cdots \smile a_n$ and a_{n+1} are comparable, then $(a_1 \smile a_2 \smile \cdots \smile a_n) \smile a_{n+1} = a_1 \smile a_2 \smile \cdots \smile a_n$. If $a_1 \smile a_2 \smile \cdots \smile a_n$ and a_{n+1} are incomparable, then $(a_1 \smile a_2 \smile \cdots \smile a_n) \smile a_{n+1}$ covers $a_1 \smile a_2 \smile \cdots \smile a_n$.*

Proof. Assume L is an atomic lattice and let $a_1, a_2, \ldots, a_{n+1} \in \mathrm{A}(L)$ be a finite sequence of atoms. For convenience, let $a = a_1 \smile a_2 \smile \cdots \smile a_n$.

Suppose a and a_{n+1} are comparable. If $a_{n+1} \leq a$, then $a \smile a_{n+1} = a$. If $a \leq a_{n+1}$, then $a = a_{n+1}$ since only $\widehat{0}$ and a_{n+1} satisfy this relationship and $a \neq \widehat{0}$. Thus again $a_{n+1} \leq a$ so that $a \smile a_{n+1} = a$.

Now suppose a and a_{n+1} are incomparable. Then $a < a \smile a_{n+1}$. Since any $z \in [a, a_{n+1}]$ must be the join of at least $\{a_1, a_2, \ldots, a_n\}$ and at most $\{a_1, a_2, \ldots, a_{n+1}\}$, it follows that (a, a_{n+1}) is empty. Therefore, $a \smile a_{n+1}$ covers a. □

Corollary 3.3.1. *Let a and a_{n+1} be as in the previous lemma. Then a and a_{n+1} are incomparable if and only if $a \frown a_{n+1} = \widehat{0}$.*

Proof. If a and a_{n+1} are incomparable, then $a \frown a_{n+1} < a_{n+1}$. But the only element of L satisfying this relationship is $\widehat{0}$. If a and a_{n+1} are comparable, then it follows from the proof of the previous lemma that $a_{n+1} \leq a$. Therefore, $a \frown a_{n+1} = a_{n+1}$. □

3.4. **Semimodular Independence and Geometric Lattices.**

Theorem 3.4.1. *Let L be a semimodular lattice with rank function ρ. Given any finite sequence $x_1, x_2, \ldots, x_n \in L$, $\rho(x_1 \smile x_2 \smile \cdots \smile x_n) \leq \rho(x_1) + \rho(x_2) + \cdots + \rho(x_n)$.*

Proof. Proof by induction on the length n of the finite sequence $x_1, x_2, \ldots, x_n \in L$. Suppose $n = 1$. Then, trivially, $\rho(x_1) \leq \rho(x_1)$.

Suppose the statement of the theorem is true for some $k \in \mathbb{N}$. Given any finite sequence $x_1, x_2, \ldots, x_{k+1} \in L$, the semimodularity of L and the induction hypothesis imply $\rho((x_1 \smile x_2 \smile \cdots \smile x_k) \smile x_{k+1}) + \rho((x_1 \smile x_2 \smile \cdots \smile x_k) \frown x_{k+1}) \leq \rho(x_1 \smile x_2 \smile \cdots \smile x_k) + \rho(x_{k+1}) \leq \rho(x_1) + \rho(x_2) + \cdots + \rho(x_k) + \rho(x_{k+1})$; i.e., $\rho((x_1 \smile x_2 \smile \cdots \smile x_k) \smile x_{k+1}) \leq \rho(x_1) + \rho(x_2) + \cdots + \rho(x_k) + \rho(x_{k+1}) - \rho((x_1 \smile x_2 \smile \cdots \smile x_k) \frown x_{k+1})$. Since $\rho((x_1 \smile x_2 \smile \cdots \smile x_k) \frown x_{k+1})$ is nonnegative, the statement of the theorem is true for any finite sequence of length $k + 1$. Therefore, by mathematical induction, the theorem is true for all finite sequences in L. □

Definition 3.4.1. Let L be an upper semimodular lattice with rank function ρ. A finite sequence $x_1, x_2, \ldots, x_n \in L$ is *independent* if $\rho(x_1 \smile x_2 \smile \cdots \smile x_n) = \rho(x_1) + \rho(x_2) + \cdots + \rho(x_n)$.

Lemma 3.4.1. *Let L be an atomic, upper semimodular lattice with rank function ρ. Then $a_1, a_2, \ldots, a_n \in L$ is independent in L if and only if for all $i, j \in [n]$, if $i \neq j$, then a_i and a_j are incomparable.*

Proof. Assume L is an atomic, upper semimodular lattice with rank function ρ. Let $a_1, a_2, \ldots, a_n \in \mathrm{A}(L)$ be a finite sequence of atoms.

($\Rightarrow$) Suppose $a_1, a_2, \ldots, a_n$ is independent in L. Proceed by induction on n. If $n = 1$, then $\rho(x_1) = \rho(x_1)$. Suppose the the statement is true for some $k \in \mathbb{N}$. Semimodularity and independence imply $k + 1 = \rho(a_1 \smile a_2 \smile \cdots \smile a_{k+1}) = \rho(a_1 \smile a_2 \smile \cdots \smile a_k) + \rho(a_{k+1}) - \rho((a_1 \smile a_2 \smile \cdots \smile a_k) \frown a_{k+1}) = k + 1 - \rho((a_1 \smile a_2 \smile \cdots \smile a_k) \frown a_{k+1})$. The induction hypothesis implies $a_1, a_2, \ldots, a_k$ are pairwise incomparable. If a_{k+1} is comparable with any elements of $a_1, a_2, \ldots, a_k$, Corrolary 3.3.1 implies that $(a_1 \smile a_2 \smile \cdots \smile a_k) \frown a_{k+1} \neq \widehat{0}$, so that $\rho((a_1 \smile a_2 \smile \cdots \smile a_k) \frown a_{k+1}) \neq 0$, contradicting the independence of $a_1, a_2, \ldots, a_{k+1}$. Therefore, $a_1, a_2, \ldots, a_{k+1}$ are all pairwise incomparable. Thus the statement is true when $n = k + 1$, and therefore, by mathematical induction, true for all independent sequences of atoms.

($\Leftarrow$) Suppose the atoms $a_1, a_2, \ldots, a_n$ are all pairwise incomparable. By Lemma 3.3.1, a_{k+1} covers $a_1 \smile a_2 \smile \cdots \smile a_k$ for all $k \in [n-1]$; i.e., $\rho(a_1 \smile a_2 \smile \cdots \smile a_{k+1}) = \rho(a_1 \smile a_2 \smile \cdots \smile a_k) + 1$. It follows then that $\rho(a_1 \smile a_2 \smile \cdots \smile a_n) = n = \rho(a_1) + \rho(a_2) + \cdots + \rho(a_n)$. Therefore, the sequence $a_1, a_2, \ldots, a_n$ is independent. □

Theorem 3.4.2. *Let L be a finite upper semimodular lattice. The following conditions are equivalent:*

(1) *L is relatively complemented.*
(2) *L is atomic.*

Proof. Assume L is a finite upper semimodular lattice with rank function ρ.

($1 \Rightarrow 2$) Assume L is relatively complemented. Suppose L is not atomic. Choose $x \in L \setminus \{\widehat{0}\}$ s.t. x is not the join of atoms and $\rho(x)$ is minimal.

By our assumptions, $[\widehat{0}, x]$ is complemented. Let a be an atom of $[\widehat{0}, x]$ and let c be a complement of a in $[\widehat{0}, x]$. $c \neq x$, else $a \leq c$ so that $a \frown c = a$, contradicting the fact that c is a complement of a. Thus $c < x$, hence $\rho(c) < \rho(x)$. The minimality of $\rho(x)$ implies that c is the join of atoms. But $x = a \smile c$, contradicting the fact that x is not equal to the join of atoms. Therefore, L is atomic.

($2 \Rightarrow 1$) Assume L is atomic and let $[x, y]$ be any interval of L. Let $z \in [x, y]$. By Lemma 3.4.1, we can choose a finite independent sequence of atoms $a_1, a_2, \ldots, a_n$ that is also independent of z s.t. $z \smile (a_1 \smile a_2 \smile \cdots \smile a_n) = y$. Let $a = a_1 \smile a_2 \smile \cdots \smile a_n$. Lemma 3.4.1 now implies that $\rho(z \smile a) = \rho(z) + n$, and since $x \leq z$, $\rho(x \smile a) = \rho(x) + n$.

Let $c = x \smile a$. Then $z \smile c = z \smile (x \smile a) = (z \smile x) \smile a = z \smile a = y$. Since $x \leq z$ and $x \leq c$, $x \leq z \frown c$. Semimodularity implies $\rho(z \frown c) \leq \rho(z) + \rho(c) - \rho(z \smile c) = \rho(z) + \rho(x \smile a) - \rho(z \smile a) = \rho(z) + \rho(x) + n - (\rho(z) + n) = \rho(x)$. Thus $z \frown c = x$, proving c is the complement of z in $[x, y]$. Therefore, L is relatively complemented [3, 105-106]. □

Definition 3.4.2. A finite semimodular lattice satisfying either of the conditions of Theorem 3.4.2 is a *geometric lattice*.

4. Lattices of Partitions

In this section we will apply what we have learned so far to the lattice of partitions of an n-set, Π_n, and to the lattice of noncrossing partitions of a kn-set with blocks of cardinality divisble by k, $\mathrm{NC}_{k,n}$.

4.1. **The Lattice of Partitions of an n-Set.**

Given $n \in \mathbb{N}$, recall that Π_n is the set of all partitions of the set $[n]$, ordered by refinement; i.e., $\pi \leq \pi'$ in Π_n if and only if for all $B \in \pi$ there exists $B' \in \pi'$ s.t. $B \subseteq B'$. When writing partitions of $[n]$, it is sometimes convenient to seperate blocks by a slash (/) and elements in a block, written in ascending order, by comma (,). Thus the partition $\{\{1\},\{2,3\}\} \in \Pi_3$ is sometimes written $1/2,3$.

Clearly Π_n is finite. It also possesses an infimum and supremum, namely $\widehat{0} = 1/2/\ldots/n$ and $\widehat{1} = 1,2,\ldots,n$.

Lemma 4.1.1. *π' covers $\pi = \{B_1, B_2, \ldots, B_l\}$ in Π_n if and only if there exist distinct $i, j \in [l]$ s.t. $\pi' = (\pi \setminus \{B_i, B_j\}) \cup \{B_i \cup B_j\}$.*

Proof. ($\Rightarrow$) Assume π' covers $\pi = \{B_1, B_2, \ldots, B_l\}$ in Π_n. Suppose there does not exist distinct $i, j \in [l]$ s.t. $\pi' = (\pi \setminus \{B_i, B_j\}) \cup \{B_i \cup B_j\}$. Since $\pi < \pi'$, two cases follow:

1. There are distinct $A, B \in \pi'$ and distinct $a, b, c, d \in [l]$ s.t. $B_a \cup B_b \subseteq A$ and $B_c \cup B_d \subseteq B$. But then $\pi < (\pi \setminus \{B_a, B_b\}) \cup \{B_a \cup B_b\} < \pi'$, contradicting the fact that $(\pi, \pi') = \emptyset$.
2. There are distinct $A, B, C \in \pi'$ s.t. for some $D \in \pi$, $A \cup B \cup C \subseteq D$. But then $\pi < (\pi \setminus \{A, B\}) \cup \{A \cup B\} < \pi'$, again contradicting the fact that $(\pi, \pi') = \emptyset$.

Since both cases contradict the fact that π' covers π, there must exist distinct $i, j \in [l]$ s.t. $\pi' = (\pi \setminus \{B_i, B_j\}) \cup \{B_i \cup B_j\}$. ☐

Remark Notice that, in the previous theorem, $|\pi'| = |\pi| - 1$.

Corollary 4.1.1. *Let $\pi \leq \sigma$ in Π_n. The each block of σ is the union of blocks of π.*

Proof. Assume $\pi \leq \sigma$ in Π_n. If $\pi = \sigma$, the result is obvious. Lemma 4.1.1 implies the result when σ covers π. Suppose then that π is neither equal to or covered by σ. Let $\pi = \pi_0 < \pi_1 < \cdots < \pi_l = \sigma$ be a saturated chain of Π_n. Thus for all $i \in [l]$, π_i covers π_{i-1}. By Lemma 4.1.1, each block of π_i is the union of blocks of π_{i-1}. It then follows by induction that each block of σ is the union of blocks of π. ☐

Theorem 4.1.1. *Π_n is graded of rank $n-1$. If ρ is the rank function of Π_n and $\pi \in \Pi_n$, then $\rho(\pi) = n - |\pi|$.*

Proof. Let $C = \{\pi_0, \pi_1, \ldots, \pi_l\}$ be a maximal chain of Π_n s.t. $\pi_0 < \pi_1 < \cdots < \pi_l$. Then $\pi_0 = \widehat{0} = 1/2/\ldots/n$, and since for all $i \in [l]$, π_i covers π_{i-1}, Lemma 4.1.1 implies $l = n-1$. Therefore, Π_n is graded of rank $n-1$.

Let ρ be the rank function of Π_n. By definition, $\rho(\widehat{0}) = 0 = n - \left|\widehat{0}\right|$. It follows from Lemma 4.1.1 and induction that for all $\pi \in \Pi_n$, $\rho(\pi) = n - |\pi|$. ☐

Theorem 4.1.2. *For all $k \in [0, n-1]$, there are*

$$S(n, n-k) = \frac{1}{(n-k)!} \sum_{i=0}^{n-k} (-1)^{n-k-i} \binom{n-k}{i} i^n$$

partitions of Π_n of rank k.

Proof. By definition, $S(n,k)$, called a *Stirling number of the second kind*, is the number of partitions of an n-set into k blocks. By convention, $S(0,0) = 1$. If $k > n$, then $S(n,k) = 0$, since there is no way to partition an n-set into more nonempty blocks than there are elements. If $n \geq 1$, then the following are true:

(1) $S(n, 0) = 0$, since there is no way to partition an n-set into zero blocks.
(2) $S(n, 1) = 1$, since the only such partition is $1, 2, \ldots, n$.
(3) $S(n, 2) = 2^{n-1} - 1$. To see this, notice that this is essentially a problem of choosing which of two indistinct bins to place each of the distinct n elements without leaving a bin empty. After we have placed $n-1$ of the elements, there are two possible cases to consider:
    (a) One bin is empty. Then all $n-1$ elements were placed in the same bin. There is just one way of doing this, and we are then forced to place the nth element in the other bin.
    (b) Neither bin is empty. There are $S(n-1, 2)$ ways of doing this. The nth element can then be placed in either of the two bins. Thus there is a total of $2 \cdot S(n-1, 2)$ ways to place the n elements.

    Thus $S(n,2) = 2 \cdot S(n-1,2) + 1$. First, $S(1,2) = 0$ since $2 > 1$ and $2^{1-1} - 1 = 2^0 - 1 = 1 - 1 = 0$. If we suppose that, for some $k \in \mathbb{N}$, $S(k,2) = 2^{k-1} - 1$, then $S(k+1, 2) = 2 \cdot S(k,2) + 1 = 2(2^{k-1} - 1) + 1 = 2^k - 2 + 1 = 2^{(k+1)-1} - 1$. Therefore, by mathematical induction, $S(n,2) = 2^{n-1} - 1$.
(4) $S(n, n-1) = \binom{n}{2}$. To see this, notice first that all the bins must be nonempty. So after placing all n elements, $n-2$ of the bins will contain just one element, while the other bin will contain two elements. The number of ways of doing this is just the number of ways of choosing two elements from n; i.e, $\binom{n}{2}$ ways.
(5) $S(n, n) = 1$, since the only such partition is $1/2/\ldots/n$.

In general, placing $n-1$ of n distinct elements into k indistinct bins yields two cases:

(1) One bin is left empty. Thus the $n-1$ elements were placed in $k-1$ bins. The number of ways of doing this is $S(n-1, k-1)$. The last element must be placed in the empty bin.
(2) No bin is left empty. Thus the $n-1$ elements were place in k bins. The number of ways of doing this is $S(n-1, k)$. The last element can then be placed in any of the k bins. Thus this case yields a total of $k \cdot S(n-1, k)$ ways to place the n elements.

Therefore, $S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$. If for all $k \in \mathbb{N} \cup \{0\}$ we define $F_k(x) := \sum_{n=k}^{\infty} \frac{S(n,k)}{n!} x^n$, then

$$F_k(x) = k \sum_{n=k}^{\infty} \frac{S(n-1,k)}{n!} x^n + \sum_{n=k}^{\infty} \frac{S(n-1,k-1)}{n!} x^n.$$

Differentiating boths sides with respect to x gives

$$F_k'(x) = k\sum_{n=k}^{\infty}\frac{n\cdot S(n-1,k)}{n!}x^{n-1} + \sum_{n=k}^{\infty}\frac{n\cdot S(n-1,k-1)}{n!}x^{n-1} =$$

$$k\sum_{n=k}^{\infty}\frac{S(n-1,k)}{(n-1)!}x^{n-1} + \sum_{n=k}^{\infty}\frac{S(n-1,k-1)}{(n-1)!}x^{n-1} =$$

$$k\sum_{n=k}^{\infty}\frac{S(n,k)}{n!}x^{n} + \sum_{n=k-1}^{\infty}\frac{S(n,k-1)}{n!}x^{n} = kF_k(x) + F_{k-1}(x).$$

Now, $F_0(x) = \sum_{n=0}^{\infty}\frac{S(n,0)}{n!}x^n = S(0,0) + \sum_{n=1}^{\infty}\frac{S(n,0)}{n!}x^n = 1 + \sum_{n=1}^{\infty}\frac{0}{n!}x^n = 1 + 0 = 1$. Also, if for all $k \in \mathbb{N}\cup\{0\}$ we define $f_k(x) := \frac{1}{k!}(e^x-1)^k$, then $f_0(x) = \frac{1}{0!}(e^x-1)^0 = \frac{1}{1}\cdot 1 = 1$. Suppose that, for some $k \in \mathbb{N}$, $F_{k-1}(x) = f_{k-1}(x)$. Notice that we now have a nonhomogeneous ordinary differential equation $F_k'(x) - kF_k(x) = F_{k-1}(x) = f_{k-1}(x) = \frac{1}{(k-1)!}(e^x-1)^{k-1}$. It thus has a unique solution. Try $F_k(x) = f_k(x)$:

$$f_k'(x) - kf_k(x) = \frac{d}{dx}(\frac{1}{k!}(e^x-1)^k) - k\frac{1}{k!}(e^x-1)^k =$$

$$\frac{k}{k!}(e^x-1)^{k-1}e^x - \frac{1}{(k-1)!}(e^x-1)^k =$$

$$\frac{1}{(k-1)!}(e^x-1)^{k-1}e^x - \frac{1}{(k-1)!}(e^x-1)^{k-1}(e^x-1) =$$

$$(\frac{1}{(k-1)!}(e^x-1)^{k-1})(e^x-(e^x-1)) = f_{k-1}(x)(e^x-e^x+1) = f_{k-1}(x).$$

Therefore, by mathematical induction, $F_k(x) = \frac{1}{k!}(e^x-1)^k$ for all $k \in \mathbb{N}\cup\{0\}$.

If we now write

$$F_k(x) = \sum_{n=k}^{\infty}\frac{S(n,k)}{n!}x^n = \frac{1}{k!}(e^x-1)^k = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}e^{ix} =$$

$$\frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\sum_{n=0}^{\infty}\frac{i^n}{n!}x^n = \sum_{n=0}^{\infty}\frac{\frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n}{n!}x^n$$

and equate coefficients, we see that $S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n$. Since a partition $\pi \in \Pi_n$ of rank $k = n - |\pi|$, π has $n-k$ blocks. Therefore, there are $S(n,n-k) = \frac{1}{(n-k)!}\sum_{i=0}^{n-k}(-1)^{n-k-i}\binom{n-k}{i}i^n$ partitions of of Π_n of rank k [15, 33-34]. □

Corollary 4.1.2. *The rank-generating function for* Π_n *is*

$$\mathrm{F}(\Pi_n, x) = \sum_{k=0}^{n-1} S(n,n-k)x^k.$$

Proof. This follows directly from Theorem 4.1.2. □

Definition 4.1.1. For all $n \in \mathbb{N}$, the number $B(n) := \mathrm{F}(\Pi_n, 1)$ is called the nth *Bell number.*

Remark Notice that the nth Bell number is equal to the cardinality of Π_n.

Theorem 4.1.3. Π_n *is a geometric lattice.*

Proof. Given $\pi, \sigma \in \Pi_n$, let $\tau = \{A \cap B \mid A \in \pi, B \in \sigma, A \cap B \neq \emptyset\}$. Then $\tau \in Low(\pi, \sigma)$. Given $\upsilon \in Low(\pi, \sigma)$ and $B \in \upsilon$, there exists $P \in \pi$ and $S \in \sigma$ s.t. $B \subseteq P$ and $B \subseteq S$. Thus $B \subseteq P \cap S$. Hence $P \cap S \neq \emptyset$, and so $P \cap S \in \tau$. It follows then that $\upsilon \leq \tau$. Thus τ is the maximal element of $Low(\pi, \sigma)$, and so $\pi \frown \sigma = \tau$. Therefore, Π_n is a meet-semilattice. Since Π_n possesses a supremum, it follows by Theorem 3.1.2 that Π_n is a lattice.

It is obvious that Π_1 and Π_2 are geometric ($\Pi_1 \cong \mathbf{1}$ and $\Pi_2 \cong \mathbf{2}$). So suppose $n \geq 3$ and let $\alpha, \beta \in \mathrm{A}(\Pi_n)$ be distinct atoms of Π_n. It follows from Lemma 4.1.1 that α contains all singleton blocks except one block $A = \{a, a'\}$ which contains two elements. The same is true for β, so call its non-singleton block $B = \{b, b'\}$. Then $A \neq B$, else $\alpha = \beta$. If $A \cap B \neq \emptyset$, then $\alpha \smile \beta = (\widehat{0} \setminus \{a, a', b, b'\}) \cup \{A \cup B\}$. If $A \cap B = \emptyset$, then $\alpha \smile \beta = (\widehat{0} \setminus \{a, a', b, b'\}) \cup \{A, B\}$. In either case, the rank of $\alpha \smile \beta$ is two. Thus $\alpha \smile \beta$ covers both α and β.

Suppose $\pi = \{B_1, B_2, \ldots, B_l\} \in \Pi_n$. Define a function $\phi : [\pi, \widehat{1}] \to \Pi_l$ for all $\tau \in [\pi, \widehat{1}]$ by $\phi(\tau) = \{I \subseteq [l] \mid \exists B \in \tau \text{ s.t. } B = \cup_{i \in I} B_i\}$. Corrolary 4.1.1 implies ϕ is a well-defined isomorphism. Therefore, $[\pi, \widehat{1}] \cong \Pi_l$.

Now let σ and τ both cover π. Then $\phi(\sigma)$ and $\phi(\tau)$ both cover $\phi(\pi) = \widehat{0}$ in Π_l. Thus $\phi(\sigma) \smile \phi(\tau)$ covers both $\phi(\sigma)$ and $\phi(\tau)$. Thus $\sigma \smile \tau$ covers both σ and τ. Therefore, Π_n is upper semimodular. Corrolary 4.1.1 implies Π_n is atomic. Therefore, Π_n is geometric. □

4.2. **The Lattice of Noncrossing Partitions of an n-Set.** Given $k, n \in \mathbb{N}$, recall that $\mathrm{NC}_{k,n}$ is the subposet of all noncrossing partitions of Π_{kn}, the cardinality of whose blocks are divisble by k. Because $\mathrm{NC}_{k,n}$ is a subposet of Π_{kn}, it will adopt many of the same attributes as Π_{kn}. For instance, Lemma 4.1.1 and Corrolary 4.1.1 clearly apply to $\mathrm{NC}_{k,n}$. $\mathrm{NC}_{k,n}$ always possesses the supremum $1, 2, \ldots, kn$, but only NC_n possesses an infimum $1/2/\ldots/n$.

Theorem 4.2.1. *$\mathrm{NC}_{k,n}$ is graded of rank $n-1$. If ρ is the rank function of $\mathrm{NC}_{k,n}$ and $\pi \in \mathrm{NC}_{k,n}$, then $\rho(\pi) = t - |\pi|$.*

Proof. Let $\pi_0 < \pi_1 < \cdots < \pi_l$ be a maximal chain of $\mathrm{NC}_{k,n}$. π_0 is a minimal element of $\mathrm{NC}_{k,n}$, and so contains n blocks, each of cardinality k. It follows by Lemma 4.1.1 and induction that $l = n - 1$. Therefore, $\mathrm{NC}_{k,n}$ is graded of rank $n - 1$. If ρ is the rank function of $\mathrm{NC}_{k,n}$, then again follows by Lemma 4.1.1 and induction that for all $\pi \in \mathrm{NC}_{k,n}$, $\rho(\pi) = n - |\pi|$. □

Theorem 4.2.2. *For all $r \in [0, n-1]$, there are $\frac{1}{n}\binom{n}{n-r}\binom{kn}{n-r-1}$ partitions of $\mathrm{NC}_{k,n}$ of rank r.*

Proof. We will first prove this for the case $k = 1$, and then generalize for all k.

Given $n \in \mathbb{N}$ and $b \in [n]$, define $\sigma_b(n) = \{s_1, s_2, \ldots, s_n\}$ to be the finite sequence

$$s_1 = b, s_2 = b + 1, \ldots, s_{n-b+1} = n, s_{n-b+2} = 1, s_{n-b+3} = 2, \ldots, s_n = b - 1.$$

Given $X \in \mathrm{NC}_n$, the blocks of X can be ordered relative to $\sigma_b(n)$ by letting B_1 be the block containing b and, for all $i \in [2, n]$, letting B_i be the block containing the number furthest to the left in $\sigma_b(n)$ not contained in $B_1 \cup B_2 \cup \cdots \cup B_{i-1}$.

Given $k \in \mathbb{N} \cup \{0\}$, let $(L, R_1, R_2, \ldots, R_k) \in (B_n \setminus \{\emptyset\})^{k+1}$ s.t. $|L| = (\sum_{1=1}^{k} |R_i|) + 1$. Parenthesize $\sigma_b(n)$ in the following way: insert an open parenthesis to the left of every element of $\sigma_b(n)$ contained in L and a closed parenthesis to the right of every element of $\sigma_b(n)$ any time it appears in the sets $R_1, R_2, \ldots, R_k$. $\sigma_b(n)$

parenthesized in this way is denoted $\hat{\sigma}_b(n)$, and is *well-parenthesized* if it begins with an open parenthesis and, with the removal of that open parenthesis, the remaining parenthesis all close. This leads to an important lemma:

Lemma 4.2.1. *Let* $(L, R_1, R_2, \dots, R_k) \in (B_n \setminus \{\emptyset\})^{k+1}$ *s.t.* $|L| = (\sum_{1=1}^{k} |R_i|) + 1$. *Then given* $n \in \mathbb{N}$, *there exists a unique* $b \in [n]$ *s.t.* $\hat{\sigma}_b(n)$ *is well-parenthesized.*

Proof. Assume $n \in \mathbb{N}$. We proceed by induction on the cardinality of L. If $|L| = 1$, then $L = \{x\}$ for some $x \in [n]$. Then k must be equal to 0, so there are no subsets R. Therefore, $b = x$.

Suppose the theorem is true for some $l \in \mathbb{N}$. Let $(L, R_1, R_2, \dots, R_k) \in B_n^{k+1}$ s.t. $|L| = l + 1$. Choose $x \in L$ and, for some $i \in [k]$, $y \in R_i$ s.t. the block $(x, \dots, y)$ contains no internal parentheses. If we remove x from L and y from R_i, we get a k- or $k + 1$-tuple s.t. $|L| = l$. The induction hypothesis provides a unique b s.t. $\hat{\sigma}_b(n)$ is well-parenthesized with respect to the new tuple. Let $r, t \in [n]$ s.t. $s_r = x$ and $s_t = y$ with respect to $\hat{\sigma}_b(n)$. It follows from our choice of x and y ($(x \dots y)$ contained no internal parentheses) that $r \leq t$. Let $\hat{\sigma}_b(n)'$ be $\hat{\sigma}_b(n)$ with an open parenthesis to the left of x and a closed parenthesis to the right of y. The above discussion guarantees $\hat{\sigma}_b(n)'$ is well-parenthesized. Therefore, b is a number s.t. $\hat{\sigma}_b(n)$ with respect to $(L, R_1, R_2, \dots, R_k)$ is well-parenthesized. Suppose b' is another number s.t. $\hat{\sigma}_{b'}(n)$ with respect to $(L, R_1, R_2, \dots, R_k)$ is well-parenthesized, then the removal of x and y again implies $b' = b$ [2]. □

$\hat{\sigma}_b(n)$ is associated with a noncrossing partition of Π_n as follows. Add a right parenthesis to at the end of $\hat{\sigma}_b(n)$. If a substring of $\hat{\sigma}_b(n)$ is enclosed by parentheses and contains no internal parentheses, then remove the substring and the parentheses and call that a block. Then perform the same procedure on the new string. Continue this until the string is empty. The well-parenthesizedness of $\hat{\sigma}_b(n)$ ensures that the chosen blocks will form a partition of $[n]$. The way we chose the blocks guarantees the partition is noncrossing.

Define, by the previous lemma, a function ϕ from all pairs $(L, R) \in B_n^2$ s.t. $|L| = |R| + 1 = k$ into all pairs $(X, b) \in \mathrm{NC}_n \times [n]$ s.t. X has k blocks. The lemma guarantees that ϕ is injective. Given $(X, b) \in \mathrm{NC}_n \times [n]$, order the blocks of X with respect to σ_b, and then order the elements of each block as they appear in σ_b. Let L be the first elements of these blocks, and R the last elements of all but the first block. Therefore, ϕ is surjective, and so bijective.

There are $\binom{n}{k}\binom{n}{k-1}$ such pairs (L, R). Since this number is equal to the number of such pairs (X, b) times n, we see that the number of noncrossing partitions of $[n]$ with k blocks is $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. Since a noncrossing partition of $[n]$ with k has rank $n-k$, it follows that there are $\frac{1}{n}\binom{n}{n-k}\binom{n}{n-k-1}$ noncrossing partitions of $[n]$ of rank k [6, 172-173].

For the generalization, refer to [6, 175-176]. □

Theorem 4.2.3. *The rank-generating function for* $\mathrm{NC}_{k,n}$ *is*

$$\mathrm{F}(\mathrm{NC}_{k,n}, x) = \sum_{i=0}^{n-1} \frac{1}{n}\binom{n}{n-i}\binom{kn}{n-i-1} x^i$$

Proof. This follows immediately from Theorem 4.2.2. □

Remark Note that $\sum_{i=0}^{n-1} \frac{1}{n}\binom{n}{n-i}\binom{n}{n-i-1} = \sum_{i=1}^{n} \frac{1}{n}\binom{n}{i}\binom{n}{i-1} = \frac{1}{n+1}\binom{2n}{n} = C_n$, the nth Catalan number. Thus $C_n = \mathrm{F}(\mathrm{NC}_n, 1)$

Remark Since, for all $k \neq 1$, $\mathrm{NC}_{k,n}$ is finite and does not possess an infimum, it follows that $\mathrm{NC}_{k,n}$ is not a lattice. However, given $n \in \mathbb{N}$, NC_n does possess an infimum. In fact, NC_n is a lattice, called the lattice of noncrossing partitions of an n-set.

Theorem 4.2.4. NC_n *is a geometric sublattice of* Π_n.

Proof. Given $\pi, \sigma \in \mathrm{NC}_n$, let $\tau = \{A \cap B \mid A \in \pi, B \in \sigma, A \cap B \neq \emptyset\}$. Clearly τ is noncrossing, since a crossing in τ would cause a crossing in π or σ. Hence $\tau \in Low(\pi, \sigma)$. Given $\upsilon \in Low(\pi, \sigma)$ and $B \in \upsilon$, there exists $P \in \pi$ and $S \in \sigma$ s.t. $B \subseteq P$ and $B \subseteq S$. Thus $B \subseteq P \cap S$. Hence $P \cap S \neq \emptyset$, and so $P \cap S \in \tau$. It follows then that $\upsilon \leq \tau$. Thus τ is the maximal element of $Low(\pi, \sigma)$, and so $\pi \frown \sigma = \tau$. Therefore, NC_n is a meet-semilattice. Since NC_n possesses a supremum, it follows by Theorem 3.1.2 that NC_n is a lattice.

It is obvious that NC_1 and NC_2 are geometric ($\mathrm{NC}_1 \cong \Pi_1$ and $\mathrm{NC}_2 \cong \Pi_2$). So suppose $n \geq 3$ and let $\alpha, \beta \in \mathrm{A}(\mathrm{NC}_n)$ be distinct atoms of NC_n. It follows from Lemma 4.1.1 that α contains all singleton blocks except one block $A = \{a, a'\}$ which contains two elements. The same is true for β, so call its non-singleton block $B = \{b, b'\}$. Then $A \neq B$, else $\alpha = \beta$. If $A \cap B \neq \emptyset$, then $\alpha \smile \beta = (\widehat{0} \setminus \{a, a', b, b'\}) \cup \{A \cup B\}$. If $A \cap B = \emptyset$, then $\alpha \smile \beta = (\widehat{0} \setminus \{a, a', b, b'\}) \cup \{A, B\}$. In either case, the rank of $\alpha \smile \beta$ is two. Thus $\alpha \smile \beta$ covers both α and β.

Suppose $\pi = \{B_1, B_2, \ldots, B_l\} \in \mathrm{NC}_n$. Define a function $\phi : [\pi, \widehat{1}] \to \mathrm{NC}_l$ for all $\tau \in [\pi, \widehat{1}]$ by $\phi(\tau) = \{I \subseteq [l] \mid \exists B \in \tau \text{ s.t. } B = \cup_{i \in I} B_i\}$. Corrolary 4.1.1 implies ϕ is a well-defined isomorphism. Therefore, $[\pi, \widehat{1}] \cong \mathrm{NC}_l$.

Now let σ and τ both cover π. Then $\phi(\sigma)$ and $\phi(\tau)$ both cover $\phi(\pi) = \widehat{0}$ in NC_l. Thus $\phi(\sigma) \smile \phi(\tau)$ covers both $\phi(\sigma)$ and $\phi(\tau)$, implying $\sigma \smile \tau$ covers both σ and τ. Therefore, NC_n is upper semimodular. Corrolary 4.1.1 implies NC_n is atomic. Therefore, NC_n is geometric. Since NC_n is atomic, it follows that NC_n is closed under joins and meets. Therefore, NC_n is a sublattice of Π_n. □

4.3. **Meanders as a subposet of** $\mathrm{NC}_n \times \mathrm{NC}_n^*$**.** By Theorem 4.2.4 and Lemma 3.1.2, $\mathrm{NC}_n \times \mathrm{NC}_n^*$ is a lattice. It can be shown that NC_n is self-dual [14, 196-197]. Thus NC_n^* must be geometric as well. Clearly the product of two geometric lattices is again a geometric lattice. Thus $\mathrm{NC}_n \times \mathrm{NC}_n^*$ is a geometric lattice. By Corrolary 2.3.1, $\mathrm{F}(\mathrm{NC}_n \times \mathrm{NC}_n^*, x) = \mathrm{F}(\mathrm{NC}_n, x)\mathrm{F}(\mathrm{NC}_n^*, x) = [\mathrm{F}(\mathrm{NC}_n, x)]^2$ since NC_n is self-dual. Therefore, $\mathrm{F}(\mathrm{NC}_n \times \mathrm{NC}_n^*, x) = [\sum_{i=0}^{n-1} \frac{1}{n}\binom{n}{n-i}\binom{n}{n-i-1} x^i]^2 =$

$$\sum_{i=0}^{2(n-1)} \Big[\sum_{j=0}^{i} \frac{1}{n}\binom{n}{n-j}\binom{n}{n-j-1}\frac{1}{n}\binom{n}{n-(i-j)}\binom{n}{n-(i-j)-1}\Big] x^i =$$

$$\sum_{i=0}^{2(n-1)} \Big[\frac{1}{n^2}\sum_{j=0}^{i} \binom{n}{n-j}\binom{n}{n-j-1}\binom{n}{n-i+j}\binom{n}{n-i+j-1}\Big] x^i$$

so that there are $\frac{1}{n^2}\sum_{j=0}^{i}\binom{n}{n-j}\binom{n}{n-j-1}\binom{n}{n-i+j}\binom{n}{n-i+j-1}$ elements of $\mathrm{NC}_n \times \mathrm{NC}_n^*$ of rank i.

An element $\mathcal{P} := (P_+, P_-) \in \mathrm{NC}_n \times \mathrm{NC}_n^*$ is *connected* if the graph of $\mathcal{P}$, denoted $\Gamma_{\mathcal{P}}$, is connected [8, 9-10]. Any element of $\mathrm{NC}_n \times \mathrm{NC}_n^*$ is a *system of closed meanders* of order n, and any connected element of $\mathrm{NC}_n \times \mathrm{NC}_n^*$ is a *closed meander* (or *meander* for short) of order n [8, 10-11]. We denote the lattice of systems of meanders by S_n (i.e., $\mathrm{S}_n \cong \mathrm{NC}_n \times \mathrm{NC}_n^*$) and the subposet of S_n of all meanders of order n by M_n.

Little is known about M_n. It is known that M_n is a graded [8, 13-15], self-dual [7] poset. Unfortunately, M_n is not a lattice for $n \geq 4$, which has made deriving its rank-generating function a centuries-old problem.

5. Distributive Lattices

We now consider a class of lattices of the upmost combinatorial importance.

5.1. **Distributive Lattices.**

Definition 5.1.1. Let L be a lattice. L is *distributive* if $\smile$ and $\frown$ distribute over each other; i.e., for all $x, y, z \in L$

(1) $x \smile (y \frown z) = (x \smile y) \frown (x \smile z)$.
(2) $x \frown (y \smile z) = (x \frown y) \smile (x \frown z)$.

Theorem 5.1.1. *Every lattice satisfying condition* 1 *or* 2 *of Definition 5.1.1 is distributive.*

Proof. Assume L is a lattice. Suppose L satisfies condition 1 of Definition 5.1.1. Given $x, y, z \in L$, condition 1 and Theorem 3.1.1 imply $(x \frown y) \smile (x \frown z) = ((x \frown y) \smile x) \frown ((x \frown y) \smile z) = x \frown ((x \smile z) \frown (y \smile z)) = (x \frown (x \smile z)) \frown (y \smile z) = x \frown (y \smile z)$; i.e., condition 2 of Definition 5.1.1 is true. Therefore, L is distributive.

Now suppose L is a lattice satisfying condition 2. Then condition 2 and Theorem 3.1.1 imply $(x \smile y) \frown (x \smile z) = ((x \smile y) \frown x) \smile ((x \smile y) \frown z) = x \smile ((x \frown z) \smile (y \frown z)) = (x \smile (x \frown z)) \smile (y \frown z) = x \smile (y \frown z)$; i.e., condition 1 is true. Therefore, L is distributive. □

Theorem 5.1.2. *Every distributive lattice is modular.*

Proof. Assume L is a distributive lattice. Given $x, y, z \in L$ s.t. $x \leq z$, it follows that $x \smile z = z$. Thus $x \smile (y \frown z) = (x \smile y) \frown (x \smile z) = (x \smile y) \frown z$. Therefore, by Theorem 3.2.2, L is modular. □

Example 5.1.1. Some of the lattices we have consider thus far are distributive. For instance, given $k, n \in \mathbb{N}$, $\mathbf{n}$, B_n and D_n are distributive. However, Example 3.2.1 states Π_n and $\mathrm{NC}_{k,n}$ are not modular for $n > 2$, hence they can not be distributive for the same n.

5.2. **The Fundamental Theorem of Finite Distributive Lattices.**

Recall that for any poset P, $J(P)$ is the poset of all order ideals of P ordered by inclusion.

Theorem 5.2.1. *Let P be a poset. Then $J(P)$ is a distributive lattice.*

Proof. Assume P is a poset. In $J(P)$, let joins and meets correspond to set unions and intersections, respectively. Given $I, J \in J(P)$, let A and B be generators for I and J, respectively. Then $I \cup J = \langle A \rangle \cup \langle B \rangle = \langle A \cup B \rangle \in J(P)$ and $I \cap J = \langle A \rangle \cap \langle B \rangle = \langle A \cap B \rangle \in J(P)$. Thus $J(P)$ is a lattice. Since set unions and intersections distribute over each other, it follows that $J(P)$ is a distributive lattice. □

Definition 5.2.1. An element x of a lattice L is *join-irreducible* if whenever $y, z \in L$ and $x = y \smile z$, then $x = y$ or $x = z$. The set of all join-irreducibles of P is denoted $\mathcal{J}(P)$, while the subposet induced on $\mathcal{J}(P)$ by P is ambiguously denoted by $\mathcal{J}(P)$. Dually, x is *meet-irreducible* if whenever $y, z \in L$ and $x = y \frown z$, then $x = y$ or $x = z$.

Theorem 5.2.2. *Let P be a finite poset. $I \in J(P)$ is join-irreducible if and only if I is a principal order ideal.*

Proof. Assume P is a finite poset and $I \in J(P)$ is arbitrary.

($\Rightarrow$) Suppose I is join-irreducible. Since P is finite, I is finitely generated. Let $A \subseteq P$ be the set of generators of I. Suppose $|A| > 1$. Choose $a \in A$ and let $B = \{a\}$. Then $\langle A \setminus B \rangle \cup \langle B \rangle = \langle A \rangle = I$, contradicting the fact that I is join-irreducible. Therefore, $|A| = 1$, proving I is a principal order ideal.

($\Leftarrow$) Suppose I is a principal order ideal. Then there exists some $x \in L$ s.t. $I = \Lambda_x$. Let $J, K \in J(P)$ s.t. $J \cup K = I$. Let $B \subseteq P$ generate J and $C \subseteq P$ generate K. Then $B \cup C = \{x\}$. Thus $B \subseteq \{x\}$. Since B is nonempty, $B = \{x\}$. Therefore, $J = I$, proving I is join-irreducible. □

Corollary 5.2.1. *Let P be a finite poset. Then $P \cong \mathcal{J}(J(P))$.*

Proof. Assume P is a finite poset. Theorem 5.2.2 implies that the function $\phi : P \to \mathcal{J}(J(P))$, defined for all $x \in P$ by $\phi(x) = \Lambda_x$, is a bijection. Since $x \leq y$ if and only if $\Lambda_x \subseteq \Lambda_y$, it follows that ϕ is also isotone. Therefore, ϕ is an isomorphism, proving $P \cong \mathcal{J}(J(P))$. □

Lemma 5.2.1. *Let P and Q be finite posets. Then $J(P) \cong J(Q)$ if and only if $P \cong Q$.*

Proof. Assume P and Q are finite posets.

($\Rightarrow$) Suppose $J(P) \cong J(Q)$. Since $\mathcal{J}(J(P))$ and $\mathcal{J}(J(Q))$ are subposets of $J(P)$ and $J(Q)$, respectively, it follows that $\mathcal{J}(J(P)) \cong \mathcal{J}(J(Q))$. Corrolary 5.2.1 then implies $P \cong Q$.

($\Leftarrow$) Suppose $P \cong Q$. Corrolary 5.2.1 implies $\mathcal{J}(J(P)) \cong \mathcal{J}(J(Q))$. Since every order ideal I is the join of a finite collection of principal order ideals, it follows that $J(P) \cong J(Q)$. □

Theorem 5.2.3 (Fundamental Theorem of Finite Distributive Lattices)**.** *Let L be a finite distributive lattice. Then there exists a unique (up to isomorphism) finite poset P for which $L \cong J(P)$.*

Proof. Assume L is a finite distributive lattice. For each $x \in L$, define $I_x := \{y \in \mathcal{J}(L) \mid y \leq x\}$, considered as a subposet of $\mathcal{J}(L)$. Notice $I_x \in J(\mathcal{J}(L))$ since if $y \in I_x$ and $z \leq y$ in L, then $z \leq y \leq x$ in L; i.e., $z \in I_x$. Define a function $\phi : L \to J(\mathcal{J}(L))$ for all $x \in L$ by $\phi(x) = I_x$. Clearly, ϕ is well-defined.

Suppose that for $x, y \in L$, $I_x \neq I_y$. Then $I_x \setminus I_y \neq \emptyset$ or $I_y \setminus I_x \neq \emptyset$. Assume WLOG that there exists $z \in I_x \setminus I_y$. Then $z \leq x$ but $z \not\leq y$. This implies $x \neq y$. Thus ϕ is injective.

Given $I \in J(\mathcal{J}(L))$, let j be the join of I. Notice that $I \subseteq I_j$ and j is the join of I_j; i.e.,

$$\underset{i \in I}{\smile} i = j = \underset{i \in I_x}{\smile} i.$$

Let $w \in I_x$. It follows by distributivity that

$$\smile_{i\in I} (i \frown w) = (\smile_{i\in I} i) \frown w = (\smile_{i\in I_x} i) \frown w = \smile_{i\in I_x} (i \frown w).$$

The right-hand side of the above equation is just w since $w \in I_x$. That means one of the join-ands is $w \smile w = w$ and the rest are $\leq w$. Thus

$$\smile_{i\in I} (i \frown w) = w$$

and since w is join-irreducible, there must be some $i \in I$ s.t. $i \smile w = w$; i.e., $w \leq i$. Since I is an order ideal and $i \in I$, it follows that $w \in I$. Thus $I_x \subseteq I$, proving $I = I_x$. Therefore, ϕ is surjective, and thus bijective.

It is clear that $x \leq y$ if and only if $I_x \subseteq I_y$. Thus ϕ is isotone. This, together with the bijectivity of ϕ implies ϕ is an isomorphism. Therefore, $L \cong J(\mathcal{J}(L))$.

Suppose that P is a poset s.t. $J(P) \cong L$. Lemma 5.2.1 implies then that $P \cong \mathcal{J}(L)$, proving the uniqueness (up to isomorphism) of $\mathcal{J}(L)$ [15, 106]. □

5.3. The Rank of a Finite Distributive Lattice.

Lemma 5.3.1. *Let $I \in J(P)$ be an arbitrary order ideal. Then $I' \in J(P)$ covers I if and only if for some minimal element x of $P \setminus I$, $I' = I \cup \{x\}$.*

Proof. Assume I is an arbitrary order ideal of $J(P)$.

($\Rightarrow$) Suppose $I' \in J(P)$ covers I. Let $M = I' \setminus I$. Notice $M \in J(P)$. Let $x \in M$ be a minimal element of M. This implies that x is also a minimal element of $P \setminus I$. If $M \neq \{x\}$, then $I \cup \{x\} \in (I, I')$, contradicting the fact that I' covers I. Thus $M = \{x\}$ and $I' = I \cup \{x\}$.

($\Leftarrow$) Suppose $I' = I \cup \{x\}$, where x is a minimal element of P. Then $I' \in J(P)$. Clearly $(I, I') = \emptyset$. Since $I \subseteq I'$, I' covers I. □

Theorem 5.3.1. *If $|P| = n$, then $J(P)$ is graded of rank n. If ρ is the rank function of $J(P)$ and $I \in J(P)$, then $\rho(I) = |I|$.*

Proof. Assume $|P| = n$. By Theorem 5.1.2, we know $J(P)$ is modular, and thus graded. Let ρ be the rank function of $J(P)$.

Notice $\emptyset$ is the infimum of $J(P)$. Then $\rho(\emptyset) = 0 = |\emptyset|$. Since P is nonempty and finite, we can choose a minimal element from P and call it x_1. Then $\{x_1\} \in J(P)$ and covers $\emptyset$ in $J(P)$ by Lemma 5.3.1. Thus $\rho(\{x_1\}) = 1 = |\{x_1\}|$.

Suppose that for some $k \in [n-1]$ we have chosen elements $x_1, x_2, \ldots, x_k \in P$ s.t. for all $i \in [k-1]$, $\{x_1, x_2, \ldots, x_{i+1}\}$ covers $\{x_1, x_2, \ldots, x_i\}$ in $J(P)$ and for all $j \in [k]$, $\rho(\{x_1, x_2, \ldots, x_j\}) = |\{x_1, x_2, \ldots, x_j\}| = j$. Since $P \setminus \{x_1, x_2, \ldots, x_k\}$ is nonempty and finite, it contains a minimal element x_{k+1}. Lemma 5.3.1 implies then that $\{x_1, x_2, \ldots, x_{k+1}\}$ covers $\{x_1, x_2, \ldots, x_k\}$ in $J(P)$. Thus $\rho(\{x_1, x_2, \ldots, x_{k+1}\}) = \rho(\{x_1, x_2, \ldots, x_k\}) + 1 = |\{x_1, x_2, \ldots, x_k\}| = k + 1 = |\{x_1, x_2, \ldots, x_{k+1}\}|$.

It follows by mathematical induction that a maximal chain of $J(P)$ has length n. Therefore, $J(P)$ is graded of rank n. It also follows that for all $I \in J(P)$, $\rho(I) = |I|$. □

Corollary 5.3.1. *Given $n \in \mathbb{N}$, let $\mathcal{P}_n$ be the collection of all isomorphism classes of n-element posets and $\mathcal{D}_n$ the collection of all isomorphism classes of finite distributive lattices of rank n. Then $|\mathcal{P}_n| = |\mathcal{D}_n|$.*

Proof. Define $\phi : \mathcal{P}_n \to \mathcal{D}_n$ for all $[P] \in \mathcal{P}_n$ by $\phi([P]) = [J(P)]$. Theorem 5.3.1 assures that the domain and range are appropriate, and Lemma 5.2.1 implies that ϕ is well-defined. If we define $\psi : \mathcal{D}_n \to \mathcal{P}_n$ for all $[J(P)]$ by $\psi([J(P)]) = [\mathcal{J}(J(P))]$, then ψ is the inverse of ϕ since, by Theorem 5.2.2, $\mathcal{J}(J(P)) \cong P$. Therefore, ϕ is a bijection, proving $|\mathcal{P}_n| = |\mathcal{D}_n|$. □

Definition 5.3.1. Given $n \in \mathbb{N}$, the poset B_n is a *boolean algebra*.

Theorem 5.3.2. *Let L be a finite distributive lattice. The following conditions of L are equivalent:*

(1) *L is a boolean algebra,*
(2) *L is complemented,*
(3) *L is relatively complemented,*
(4) *L is atomic,*
(5) *$\widehat{1}$ is the join of atoms of L,*
(6) *L is geometric,*
(7) *every join-irreducible of L covers $\widehat{0}$,*
(8) *if $|\mathcal{J}(L)| = n$, then $|L| = 2^n$,*
(9) *for some $n \in \mathbb{N}$, $\mathrm{F}(L, x) = (1+x)^n$.*

Proof. Left to the interested reader (and not the slothful author). □

5.4. **Chains of a Finite Distributive Lattice.**

Theorem 5.4.1. *Let $m \in \mathbb{N}$. The following quantities are equal:*

(1) *the number of surjective isotone functions from P into $\mathbf{m}$.*
(2) *the number of chains of $J(P)$ of length m containing $\widehat{0} = \emptyset$ and $\widehat{1} = P$.*

Proof. Assume $m \in \mathbb{N}$. Let Σ be the collection of surjective isotone maps from P into $\mathbf{m}$ and χ the collection of chains of $J(P)$ of length m containing $\emptyset$ and P. Define a function $\phi : \Sigma \to \chi$ for all $\sigma \in \Sigma$ by $\phi(\sigma) = \{\sigma^{-1}(\mathbf{0}), \sigma^{-1}(\mathbf{1}), \ldots, \sigma^{-1}(\mathbf{m})\}$ (here we employ the convention that $\mathbf{0} = \emptyset$).

Notice that, for all $i \in [0, m]$, $\sigma^{-1}(\mathbf{i})$ is an order ideal of P. This follows from the fact that σ is isotone, for if $y \leq_P x$, then $\sigma(y) \leq_{\mathbf{m}} \sigma(x)$, implying $y \in \sigma^{-1}(\mathbf{i})$ if $x \in \sigma^{-1}(\mathbf{i})$. Also, $\sigma^{-1}(\mathbf{0}) = \emptyset$ and $\sigma^{-1}(\mathbf{m}) = P$, so that $\phi(\sigma)$ contains both $\emptyset$ and P.

If $j \in [0, m]$ s.t. $i \neq j$, then $\sigma^{-1}(\mathbf{i}) \neq \sigma^{-1}(\mathbf{j})$ since σ surjective. WLOG, assume $i < j$. Then $\sigma^{-1}(\mathbf{i}) \subsetneq \sigma^{-1}(\mathbf{j})$. Thus $\phi(\sigma) \in \chi$, proving ϕ is well-defined.

Let $\tau \in \Sigma$ s.t. $\phi(\sigma) = \phi(\tau)$. Then for each $i \in [m]$, $\sigma^{-1}(\mathbf{i}) \setminus \sigma^{-1}(\mathbf{i-1}) = \tau^{-1}(\mathbf{i}) \setminus \tau^{-1}(\mathbf{i-1})$; i.e., $\sigma^{-1}(i)$ and $\tau^{-1}(i)$. Therefore, $\sigma = \tau$, proving ϕ is injective.

Let $\{I_0, I_1, \ldots, I_m\} \in \chi$ s.t. $\emptyset = I_0 <_{J(P)} I_1 <_{J(P)} \cdots <_{J(P)} I_m = P$. Define $\upsilon : P \to [m]$ as follows: for all $i \in [m]$ and $x \in I_i \setminus I_{i-1}$, $\upsilon(x) := i$. Notice σ is surjective and isotone, and thus $\sigma \in \Sigma$. Also, for all $i \in [0, m]$, $\upsilon^{-1}(\mathbf{i}) = I_i$. Thus $\phi(\upsilon) = \{I_0, I_1, \ldots, I_m\}$, proving ϕ is surjective. Therefore, ϕ is bijective, proving $|\Sigma| = |\chi|$ [15, 110]. □

Definition 5.4.1. Let $|P| = n$. Then any surjective, isotone function $\sigma : P \to \mathbf{n}$ is a *linear extension of P* (or *extension of P to a total order*). The number of such functions is denoted $e(P)$.

Corollary 5.4.1. *$e(P)$ is equal to the number of maximal chains of $J(P)$.*

Proof. Theorem 5.3.1 implies that if $|P| = n$, then $J(P)$ is graded of rank n. It follows then from Theorem 5.4.1 that the number of chains of $J(P)$ of length n (i.e., the the number of maximal chains of $J(P)$) is equal to the number of linear extensions of P. This number is $e(P)$. $\square$

Remark Stanley claims [15, 110] that $e(P)$ is "probably the single most useful number for measuring the 'complexity' of P."

6. A Useful Algebra Review

The following is a useful review of some of the algebraic structures and theories we will need.

6.1. Rings, Fields and R-Algebras.

We review some definitions and results from ring theory.

Definition 6.1.1. A *ring* is an ordered triple $(R, +, \cdot)$, denoted ambiguously by R, consisting of a set R and two binary opeartions $+$ and $\cdot$ (called *addition* and *multiplication*, respectively) on R satisfying the following three properties:

(1) $(R, +)$ is an abelian group (the additive identity is denoted 0).
(2) multiplication is associative; i.e., for all $a, b, c \in R$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
(3) multiplication is right and left distributive over addition; i.e., for all $a, b, c \in R$, $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ and $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

The ring R is *commutative* if multiplication is commutative; i.e., for all $a, b \in R$, $a \cdot b = b \cdot a$. R is said to have an *identity* (or *contain a* 1) if there is an element $1 \in R$ s.t. for all $a \in R$, $1 \cdot a = a \cdot 1 = a$.

Remark Since $(R, +)$ is a group, every $r \in R$ will possess a unique additive inverse, denoted $-r$. If it is ambiguous as to which ring $+$, $\cdot$, 0 or 1 belong, we write instead $+_R$, $\cdot_R$, 0_R or 1_R, respectively.

Lemma 6.1.1. *If R is a ring, then for all $r \in R$, $r \cdot 0 = 0 \cdot r = 0$.*

Proof. Assume R is a ring. Given $r \in R$, $r \cdot 0 = r \cdot (0 + 0) = (r \cdot 0) + (r \cdot 0)$. Adding $-(r \cdot 0)$ to both sides yields $r \cdot 0 = 0$. Similarly, $0 \cdot r = 0$. $\square$

Definition 6.1.2. A *ring homomorphism* from a ring R into a ring A is a function $\varphi : R \to A$ s.t. for all $a, b \in R$, $\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$. If φ is also a bijection, then φ is a *ring isomorphism*, and R and A are *isomorphic*, denoted $R \cong A$.

Remark When it is understood that φ is a homomorphism of rings, we will simply call φ a *homomorphism*.

Warning! A homomorphism $\varphi : R \to A$ from a ring R into a ring A does not necessarily map 1_R to 1_A. For instance, φ could send every element of R to 0_A. This is indeed a homomorphism since for all $r, s \in R$, $0_A = \varphi(r+s) = \varphi(r)+\varphi(s) = 0_A + 0_A = 0_A$ and $0_A = \varphi(r \cdot s) = \varphi(r) \cdot \varphi(s) = 0_A \cdot 0_A = 0_A$.

Lemma 6.1.2. *Let $\varphi : R \to A$ be a homomorphism from a ring R into a ring A. Then $\varphi(0_R) = 0_A$ and for all $r \in R$, $\varphi(-r) = -\varphi(r)$.*

Proof. Assume $\varphi : R \to A$ is a homomorphism from a ring R into a ring A. Then $\varphi(0_R) = \varphi(0_R + 0_R) = \varphi(0_R) + \varphi(0_R)$. Therefore, $\varphi(0_R) = 0_A$. Hence for any $r \in R$, $0_A = \varphi(0_R) = \varphi(r + (-r)) = \varphi(r) + \varphi(-r)$. Therefore, $-\varphi(r) = \varphi(-r)$. $\square$

Definition 6.1.3. A *subring* of a ring R is a subgroup of R that is closed under multiplication.

Theorem 6.1.1. *Let $\varphi : R \to A$ be a homomorphism from a ring R into a ring A. Then $\varphi(R)$ is a subring of A.*

Proof. Assume $\varphi : R \to A$ is a homomorphism from a ring R into a ring A. Given $x, y \in \varphi(R)$, there exist $a, b \in R$ s.t. $\varphi(a) = x$ and $\varphi(b) = y$. It follows by Lemma 6.1.2 that $x + (-y) = \varphi(a) + (-\varphi(b)) = \varphi(a) + \varphi(-b) = \varphi(a + (-b)) \in \varphi(R)$. Thus $\varphi(R)$ is closed under addition and inverses, and hence a subgroup of A. Also, $x \cdot y = \varphi(a) \cdot \varphi(b) = \varphi(a \cdot b) \in \varphi(R)$, so $\varphi(R)$ is also closed under multiplication. Therefore, $\varphi(R)$ is a subring of A. □

Definition 6.1.4. A ring R with identity $1 \neq 0$ is a *division ring* (or *skew field*) if every nonzero element has a multiplicative inverse; i.e., for all $r \in R$ there exists $r' \in R$ s.t. $r \cdot r' = r' \cdot r = 1$. A commutative division ring is a *field*.

Lemma 6.1.3. *The multiplicative inverse of any element of a division ring is unique.*

Proof. Assume R is a division ring. Given $r \in R$, let $s, t \in R$ both be multiplicative inverses of r. Then, by defintion, $r \cdot s = 1 = r \cdot t$. Let x be either s or t. Then $x \cdot (r \cdot s) = (x \cdot r) \cdot s = 1 \cdot s = s$ and $x \cdot (r \cdot t) = (x \cdot r) \cdot t = 1 \cdot t = t$. Therefore, $s = t$, so that the multiplicative inverse of r is unique. □

Remark The multiplicative inverse of an element r in a division ring R is denoted r^{-1}.

Theorem 6.1.2. *Let $\varphi : R \to A$ be a homomorphism from a division ring R into a ring A with identity s.t $\varphi(1_R) = 1_A$. Then φ is injective.*

Proof. Assume $\varphi : R \to A$ is a homomorphism from a field R into a ring A s.t $\varphi(1_R) = 1_A$. Suppose there exists some $r \in R \setminus \{0_R\}$ s.t. $\varphi(r) = 0_A$. This implies $1_A = \varphi(1_R) = \varphi(r \cdot r^{-1}) = \varphi(r) \cdot \varphi(r^{-1}) = 0_A \cdot \varphi(r^{-1}) = 0_A$, a contradiction since R is a division ring. Thus $r = 0_R$.

Now, given $s, t \in R$, suppose $\varphi(s) = \varphi(t)$. Then $0_A = \varphi(s) + (-\varphi(t)) = \varphi(s) + \varphi(-t) = \varphi(s + (-t))$. The above result implies $s + (-t) = 0_A$; i.e., $s = t$. Therefore, φ is injective. □

Definition 6.1.5. The *center* of a ring R is the set $\mathrm{Z}(R) := \{z \in R \mid \forall r \in R,\ z \cdot r = r \cdot z\}$; i.e., the set of elements of R that commute multiplicatively with every element of R.

Definition 6.1.6. Let R be a commutative ring with identity. An *R-algebra* is an ordered pair (A, φ) consisting of a ring A with identity and a ring homomorphism $\varphi : R \to A$ s.t. $\varphi(1_R) = 1_A$ and $\varphi(R) \subseteq \mathrm{Z}(A)$.

Corollary 6.1.1. *Let R be a division ring and (A, φ) an R-algebra. Then $R \cong \varphi(R)$ and R is field.*

Proof. Assume R is a division ring and (A, φ) is an R-algebra. By Theorem 6.1.2, φ is injective. Thus $R \cong \varphi(R)$, so that $\varphi(R)$ is a division ring. Since $\varphi(R) \subseteq \mathrm{Z}(A)$, $\varphi(R)$ is commutative division ring; i.e., $\varphi(R)$ is a field. Therefore, R is a field. □

Remark In the case above, we say that A is an *algebra over* R. It is the same as saying that A contains the field R in its center and the identity of R and A are the same.

6.2. **Modules and Vector Spaces.** We review the definitions of a module and a vector space and give an example of a vector space we will use in the next section.

Definition 6.2.1. Let R be a ring. A *left R-module* (or a *left module over R*) is an ordered triple $(V, \oplus, \odot)$, ambiguously denoted V, consisting of a set V and two operations $\oplus$ and $\odot$ satisfying the following properties:

(1) $(V, \oplus)$ is an abelian group.
(2) $\odot : R \times V \to V$, where for all $(r, v) \in R \times V$, $r \odot v := \odot((r, v))$, is an action of R on V which satisfies:
  (a) for all $r, s \in R$ and $v \in V$, $(r \cdot s) \odot v = r \odot (s \odot v)$.
  (b) for all $r, s \in R$ and $v \in V$, $(r + s) \odot v = (r \odot v) \oplus (s \odot v)$.
  (c) for all $r \in R$ and $v, w \in V$, $r \odot (v \oplus w) = (r \odot v) \oplus (r \odot w)$.
  (d) if R has an identity, then for all $v \in V$, $1_R \odot v = v$, (in which case V is sometimes called a *unital left R-module*).

A *right R-module* (or a *right module over R*) is defined analogously.

Remark If R is commutative, then a left R-module V can be made into a right R-module by defining $r \odot v = v \odot r$ for all $r \in R$ and $v \in V$.

Definition 6.2.2. A module over R is a *vector space* if R is a field.

Theorem 6.2.1. *Let K be a field, S be a set, and V the collection of all functions from S into K. For all $f, g \in V$, $s \in S$ and $k \in K$, define $(f \oplus g)(s) := f(s) + g(s)$ and $(f \odot g)(s) := f(s) \cdot g(s)$. Then $(V, \oplus, \odot)$ is a vector space over K.*

Proof. Assume K, S, V, $\oplus$ and $\odot$ are as in the conditions of the theorem. Let f, g, h be arbitrary functions in V and $s \in S$ be an arbitrary element of S. We prove first that $(V, \oplus)$ is a group.

(1) (V is closed under $\oplus$) Since K is closed under addition, $(f \oplus g)(s) = f(s) + g(s) \in K$, so that V is closed under $\oplus$.
(2) ($\oplus$ is associative) Since addition in K is associative, $[(f \oplus g) \oplus h](s) = (f \oplus g)(s) + h(s) = (f(s) + g(s)) + h(s) = f(s) + (g(s) + h(s)) = f(s) + (g \oplus h)(s) = [f \oplus (g \oplus h)](s)$. So $\oplus$ is associative.
(3) (V has an identity) Let 0_V denote the function sending every element of S to 0_K. Then $(f \oplus 0_V)(s) = f(s) + 0_V(s) = f(s) + 0_K = f(s) = 0_K + f(s) = 0_V(s) + f(s) = (0_V \oplus f)(s)$. Thus 0_V is an identity for V under $\oplus$.
(4) (V is close under inverses) Let $-f$ be the function in V that sends every element of S to the additive inverse of f evaluated at the element. Then $(f \oplus (-f))(s) = f(s) + (-f(s)) = 0_K = 0_V(s)$. Thus $-f$ is the $\oplus$-inverse of f.

Therefore, $(V, \oplus)$ is a group. Since $(K, +)$ is an abelian group, $(f \oplus g)(s) = f(s) + g(s) = g(s) + f(s) = (g \oplus f)(s)$. Therefore, $(V, \oplus)$ is an abelian group.

Let $k, l \in K$ be arbitrary elements of K. Since multiplication in K is associative, $[(k \cdot l) \odot f](s) = (k \cdot l) \cdot f(s) = k \cdot (l \cdot f(s)) = k \cdot (l \odot f)(s) = [k \odot (l \odot f)](s)$. Since multiplication is right distributive, $[(k + l) \odot f](s) = (k + l) \cdot f(s) = (k \cdot f(s)) + (l \cdot f(s)) = (k \odot f)(s) + (l \odot f)(s) = [(k \odot f) \oplus (l \odot f)](s)$. Since multiplication is also left distributive, $[k \odot (l \oplus f)](s) = k \cdot (l \oplus f)(s) = k \cdot (l + f(s)) = (k \cdot l) + (k \cdot f(s)) =$

$(k \cdot l) + (k \odot f)(s) = [(k \cdot l) \oplus (k \odot f)](s)$. Also, $[1_K \odot f](s) = 1_K \cdot f(s) = f(s)$. Therefore, $(V, \oplus, \odot)$ is a vector space over K. □

6.3. **Tensor Products.**

Definition 6.3.1. Let R be a ring. A left (right) R-module V is *free* on the subset $S \subseteq V$ if for all $v \in V \setminus \{0_V\}$ there exist $n \in \mathbb{N}$ and unique elements $r_1, r_2, \ldots, r_n \in R \setminus \{0_R\}$ and unique elements $s_1, s_2, \ldots, s_n \in S$ s.t. $v = \sum_{i=0}^{n} r_i s_i$ $(\sum_{i=0}^{n} s_i r_i)$.

Remark It can be shown [5, 335-336] that a free R-module on the set S is unique, up to isomorphism.

Definition 6.3.2. Let S be a set. The *free abelian group* on S is the free $\mathbb{Z}$-module over S.

Definition 6.3.3. Let R be a ring, V a left R-module, W a right R-module and G the free abelian group on the set $V \times W$. Let H be the subgroup of G generated by all elements of the forms:

(1) $(v + v', w) - (v, w) - (v', w)$,
(2) $(v, w + w') - (v, w) - (v, w')$,
(3) $(vr, w) - (v, rw)$.

The quotient group G/H is the *tensor product* of V and W over R, denoted $V \otimes_R W$. The cosets of $V \otimes_R W$ are called *tensors* and, given $v \in V$ and $w \in W$, $v \otimes w$ denotes the tensor of $V \otimes_R W$ containing (v, w).

7. The Incidence Algebra of a Locally Finite Poset

In this section we introduce an algebraic structure for locally finite posets that is useful in answering many of the combinatorial questions associated with such poset.

7.1. **The Incidence Algebra.**

Throughout this section we assume that P is a locally finite poset and K is a field of characteristic zero.

Definition 7.1.1. Let $V(\mathrm{Int}(P), K)$ denote the vector space of all functions from $\mathrm{Int}(P)$ into K. Define *convolution*, denoted $*$, for all functions $f, g \in V(\mathrm{Int}(P), K)$ and $[x, y] \in \mathrm{Int}(P)$ by $(f * g)([x, y]) := \sum_{x \leq z \leq y} f([x, z]) \cdot g([z, y])$.

Remark Notice that since P is locally finite, the number of summands in the above sum is finite. Therefore, convolution is well-defined. For all $f \in I(P)K$ and $k \in \mathbb{N}$, denote the convolution of f with itself k times by f^k.

Theorem 7.1.1. *$(V(\mathrm{Int}(P), K), \oplus, *)$ is an algebra over K.*

Proof. Theorem 6.2.1 implies that $(V, \oplus)$ is an abelian group. Let $f, g, h \in V(\mathrm{Int}(P), K)$ and $[x, y] \in \mathrm{Int}(P)$ be arbitrary.

(1) (convolution is associative) Since the number of summands is finite,

$$[(f * g) * h]([x, y]) = \sum_{x \leq z \leq y} (f * g)([x, z]) \cdot h([z, y]) =$$

$$\sum_{x \leq z \leq y} [\sum_{x \leq w \leq z} f([x, w]) \cdot g([w, z])] \cdot h([z, y]) =$$

$$\sum_{x\le w\le y} f([x,w])\cdot[\sum_{w\le z\le y} g([w,z])\cdot h([z,y])] =$$

$$\sum_{x\le w\le y} f([x,w])\cdot(h*g)([w,y]) = [f*(g*h)]([x,y]).$$

(2) (convolution is left distributive) Since multiplication in K is left distributive,

$$[f*(g\oplus h)]([x,y]) = \sum_{x\le z\le y} f([x,z])\cdot(g\oplus h)([z,y]) =$$

$$\sum_{x\le z\le y} f([x,z])\cdot[g([z,y])+h([z,y])] =$$

$$\sum_{x\le z\le y} [f([x,z])\cdot g([z,y])]+[f([x,z])\cdot h([z,y])] =$$

$$\sum_{x\le z\le y} [f([x,z])\cdot g([z,y])]+\sum_{x\le z\le y} [f([x,z])\cdot h([z,y])] =$$

$$(f*g)([x,y])+(f*h)([x,y]) = [(f*g)\oplus(f*h)]([x,y]).$$

(3) (convolution is right distributive) Since multiplication in K is right distributive,

$$[(g\oplus h)*f]([x,y]) = \sum_{x\le z\le y} (g\oplus h)([x,z])\cdot f([z,y]) =$$

$$\sum_{x\le z\le y} [g([x,z])+h([x,z])]\cdot f([z,y]) =$$

$$\sum_{x\le z\le y} [g([x,z])\cdot f([z,y])]+[h([x,z])\cdot f([z,y])] =$$

$$\sum_{x\le z\le y} [g([x,z])\cdot f([z,y])]+\sum_{x\le z\le y} [h([x,z])\cdot f([z,y])] =$$

$$(g*f)([x,y])+(h*f)([x,y]) = [(g*f)\oplus(h*f)]([x,y]).$$

(4) (convolution identity) Let $\delta\in V(\mathrm{Int}(P),K)$ be defined for all $[x,y]\in\mathrm{Int}(P)$ by

$$\delta([x,y]) := \begin{cases}1, & \text{if } x=y,\\ 0, & \text{if } x\ne y.\end{cases}$$

Then $(f*\delta)([x,y]) = \sum_{x\le y\le z} f([x,z])\cdot\delta([z,y]) = f([x,y])$. Also, $(\delta*f)([x,y]) = \sum_{x\le y\le z}\delta([x,z])\cdot f([z,y]) = f([x,y])$. Therefore, δ is an identity for $V(\mathrm{Int}(P),K)$ under convolution.

Therefore, $(V(\mathrm{Int}(P),K),\oplus,*)$ is a ring with identity.

Let $\varphi: K\to V(\mathrm{Int}(P),K)$ be defined for all $k\in K$ by $\varphi(k) := \delta_k := k\odot\delta$. Given $k,l\in K$, $\varphi(k+l) = \delta_{k+l} = (k+l)\odot\delta = (k\odot\delta)\oplus(l\odot\delta) = \delta_k\oplus\delta_l = \varphi(k)\oplus\varphi(l)$. Also, $\varphi(k\cdot l) = \delta_{k\cdot l} = (k\cdot l)\odot\delta = (k\odot\delta)*(l\odot\delta) = \delta_k*\delta_l = \varphi(k)*\varphi(l)$. Thus φ is a ring homomorphism.

Since multiplication in K is commutative, it follows from the above that $\delta_k*\delta_l = (k\cdot l)\odot\delta = (l\cdot k)\odot\delta = \delta_l*\delta_k$, so that $\varphi(K)\subseteq \mathrm{Z}(V(\mathrm{Int}(P),K))$. Since $\delta_{1_K} = 1_K\odot\delta = \delta$, φ maps 1_K to $1_{V(\mathrm{Int}(P),K)}$. Therefore, $(V(\mathrm{Int}(P),K),\varphi)$ is a K-algebra. Since K is a field, $V(\mathrm{Int}(P),K)$ is an algebra over K. □

Definition 7.1.2. Let P be a locally finite poset and K a field. The *incidence algebra* of P over K, denoted $I(P, K)$, is the algebra $(V(\mathrm{Int}(P), K), \oplus, *)$ over K.

Remark For our current purposes it will suffice to let $K = \mathbb{C}$, in which case we define $I(P) := I(P, \mathbb{C})$.

Theorem 7.1.2. *Let $f \in I(P)$. The following conditions are equivalent:*

(1) *f has a left inverse,*
(2) *f has a right inverse,*
(3) *f has a two-sided inverse,*
(4) *for all $x \in P$, $f([x, x]) \neq 0$.*

If f has any inverse, then it is the unique two-sided inverse of f.

Proof. Assume $f \in I(P)$.

($1 \Leftrightarrow 4$) Suppose $g \in I(P)$ is a left inverse of f; i.e., $g * f = \delta$. This is true if and only if

$$g([x, y]) = \begin{cases} f([x, x])^{-1}, & \text{if } x = y \\ -f([y, y])^{-1}[\sum_{x \le z < y} g([x, z])f([z, y])], & \text{if } x < y. \end{cases}$$

(The second case is due to the fact that $(g * f)([x, y]) = \sum_{x \le z < y} g([x, z]f([z, y]) + g([x, y])f([y, y]))$. Thus g exists if and only if for all $x \in P$, $f([x, x]) \neq 0$.

($2 \Leftrightarrow 4$) Suppose $h \in I(P)$ is a right inverse of f. Similar to the above argument,

$$h([x, y]) = \begin{cases} f([x, x])^{-1}, & \text{if } x = y \\ -f([x, x])^{-1}[\sum_{x < z \le y} f([x, z])h([z, y])], & \text{if } x < y, \end{cases}$$

so that h exists if and only if for all $x \in P$, $f([x, x]) \neq 0$.

($3 \Leftrightarrow 4$) A two-sided inverse is necessarily a left and right inverse. Thus this result follows from the previous arguments.

Suppose that g is a left inverses of f. Then the theorem implies the existence of a right inverse h. Hence $g * f = \delta = f * h$. The theorem also provides a two-sided inverse f' of f. Then $g = g * \delta = g * (f * f') = (g * f) * f' = \delta * f' = f' = f' * \delta = f' * (f * h) = (f' * f) * h = \delta * h = h$. Therefore, $g = f' = h$, proving that any inverse is two-sided and unique [15, 114]. □

7.2. **Some Functions of the Incidence Algebra.** Throughout this subsection we will assume P is a locally finite poset. Of all the functions contained in $I(P)$, there are a few of particular interest.

7.2.1. *Delta Function.* We have already encountered the *delta function* δ, defined for all $[x, y] \in \mathrm{Int}(P)$ be

$$\delta([x, y]) := \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x < y. \end{cases}$$

Recall that δ is the identity in $I(P)$. Also recall that $\delta_z = z \odot \delta$ for all $k \in \mathbb{C}$.

7.2.2. *Zeta and Chain Functions.* Another is the *zeta function* ζ. It is defined for all $[x, y] \in \mathrm{Int}(P)$ by $\zeta([x, y]) := 1$. Then for all $[x, y] \in \mathrm{Int}(P)$, $\zeta^2([x, y]) = \sum_{x \le z \le y} \zeta([x, z]) \cdot \zeta([z, y]) = \sum_{x \le z \le y} 1 \cdot 1 = \sum_{x \le z \le y} 1 = |[x, y]|$. Therefore, $\zeta^2([x, y])$ counts the numbers of elements in $[x, y]$.

Notice that

$$(\zeta-\delta)([x,y]) := \begin{cases} 1-1=0 & \text{if } x=y, \\ 1-0=1 & \text{if } x<y. \end{cases}$$

Therefore, $(\zeta-\delta)^2([x,y]) = \sum_{x\leq z\leq y}(\zeta-\delta)([x,z])\cdot(\zeta-\delta)([z,y]) = \sum_{x<z<y}\zeta([x,z])\cdot\zeta([z,y]) = \sum_{x<z<y} 1$, which is equal to the number of chains of $[x,y]$ of length 2. By induction it follows that $(\zeta-\delta)^k([x,y])$ counts the number of chains of $[x,y]$ of length k. Thus $\eta := \zeta-\delta$ is the *chain function*.

A sequence of functions $f_1, f_2, f_3, \ldots$ of $I(P)$ *converges* to a function $f \in I(P)$ if for all $[x,y] \in \mathrm{Int}(P)$ there exists $N \in \mathbb{N}$ s.t. for all $n \geq N$, $f_n([x,y]) = f([x,y])$. This defines a topology on $I(P)$. Now consider

$$(\delta_2-\zeta)([x,y]) := \begin{cases} 2-1=1 & \text{if } x=y, \\ 0-1=-1 & \text{if } x<y. \end{cases}$$

By Theorem 7.1.2, $\delta_2-\zeta$ has an inverse: $(\delta_2-\zeta)^{-1} = (\delta-(\zeta-\delta))^{-1} = \sum_{k=0}^{\infty}\eta^k$, which is valid because $\sum_{k=0}^{\infty}\eta^k$ converges in $I(P)$ for all $[x,y] \in \mathrm{Int}(P)$. Therefore, because of our interpretation of η above, $(\delta_2-\zeta)^{-1}([x,y])$ counts the total number of chains of $[x,y]$ [15, 115].

7.2.3. *Lambda and Cover Functions.* Yet another important function in $I(P)$ is the *lambda function* λ defined for all $[x,y] \in \mathrm{Int}(P)$ by

$$\lambda([x,y]) := \begin{cases} 1 & \text{if } x=y \text{ or } y \text{ covers } x, \\ 0 & \text{else.} \end{cases}$$

Notice then that

$$(\lambda-\delta)([x,y]) = \begin{cases} 1 & \text{if } y \text{ covers } x, \\ 0 & \text{else.} \end{cases}$$

Thus $\kappa := \lambda-\delta$ is the *cover function*. Notice that $\kappa^2([x,y]) = \sum_{x\leq z\leq y}\kappa([x,z])\cdot\kappa([z,y])$. Since $\kappa([x,z])\cdot\kappa([z,y]) \neq 0$ only when y covers z covers x; i.e., $x<y<z$ is a saturated chain. Since $[x,y]$ is finite, this is the same as requiring $x<y<z$ to be a maximal chain. Therefore, $\kappa^2([x,y])$ counts the number of maximal chains of $[x,y]$ of length 2. By induction it follows that $\kappa^k([x,y])$ counts the number of maximal chains of $[x,y]$ of length k.

Now consider

$$(\delta-\kappa)([x,y]) := \begin{cases} 1-0=1 & \text{if } x=y, \\ 0-1=-1 & \text{if } y \text{ covers } x, \\ 0-0=0 & \text{else.} \end{cases}$$

By Theorem 7.1.2, $\delta-\kappa$ has an inverse: $(\delta-\kappa)^{-1} = \sum_{k=0}^{\infty}\kappa^k$, which is valid because $\sum_{k=0}^{\infty}\kappa^k$ converges in $I(P)$ for all $[x,y] \in \mathrm{Int}(P)$. By our interpretation of κ above, $(\delta-\kappa)([x,y])$ counts the total number of maximal chains of $[x,y]$.

7.2.4. *Möbius Function.* By Theorem 7.1.2, ζ possesses an inverse in $I(P)$. The *Möbius function* $\mu := \zeta^{-1}$. The relation $\mu * \zeta = \delta$ is equivalent to the following recursive defintion of μ for all $[x,y] \in \mathrm{Int}(P)$:

$$\mu([x,y]) := \begin{cases} 1 & \text{if } x=y, \\ -\sum_{x\leq z<y}\mu([x,z]) & \text{if } x<y, \end{cases}$$

since $(\mu * \zeta)([x, y]) = \sum_{x \leq z \leq y} \mu([x, z]) \cdot \zeta([z, y]) = \sum_{x \leq z \leq y} \mu([x, z]) \cdot 1 =$

$$\sum_{x \leq z < y} \mu([x, z]) + \mu([x, y]) = \delta([x, y]).$$

We will see in the next subsection that μ plays an important role in the algebra $I(P)$.

7.3. **Möbius Inversion Formula.** Throughout this section, assume P is a locally finite poset and let $\mathbb{C}^P$ denote the set of all functions from P into $\mathbb{C}$.

Theorem 7.3.1 (Möbius Inversion Formula)**.** *Let every principal order ideal of* P *be finite. Then for all* $f, g \in \mathbb{C}^P$ *and* $x \in P$*,* $g(x) = \sum_{y \in \Lambda_x} f(y)$ *if and only if* $f(x) = \sum_{y \in \Lambda_x} g(y)\mu([y, x])$*.*

Proof. Assume that every principal order ideal of P is finite. Then the summations in the statement of the theorem are finite, and so well-defined.

Notice that $\mathbb{C}^P$ is a vector space on which $I(P)$ acts on the right as an algebra of linear transformations by $(f\phi)(x) = \sum_{y \in \Lambda_x} f(y) \cdot \phi([x, y])$, for all $\phi \in I(P)$. Thus the statement of the theorem is simply an observation in linear algebra that $f\zeta = g$ if and only if $f = g\mu$ [15, 116]. □

Remark Of course, this theorem did not depend on $\mathbb{C}$ in the least. Thus the Möbius Inversion Formula is still true when the incidence algebra is $I(P, K)$ for some field K of characteristic zero, and K^P is the set of all functions from P into K. It should also be clear that the dual statement of the theorem, requiring finite principal dual order ideals and noticing that $I(P)$ acts on the left as an algebra of linear transformations of $\mathbb{C}^P$, is true [15, 116-117].

8. A Useful Algebraic Topology Review

The following is a brief review of some of concepts of algebraic topology we will need.

8.1. **Simplicial Complexes and Order Complexes.**

Definition 8.1.1. Let V be a set. A *simplicial complex* Δ is a collection of subsets of V s.t.:

(1) for all $v \in V$, $\{v\} \in \Delta$,
(2) for all $F \in \Delta$, if $F' \subseteq F$, then $F' \in \Delta$.

The elements of V are called *vertices* and V is called the *vertex set.* The elements of Δ are called *faces.* We require $\emptyset \in \Delta$ unless $\Delta = \emptyset$. The *dimension* of $F \in \Delta$ is the number $\dim(F) := |F| - 1$ and the *dimension* of Δ is the number $\dim(\Delta) := \max\{\dim(F) \mid F \in \Delta\}$.

Definition 8.1.2. Let P be a poset. The *order complex* of P, denoted $\Delta(P)$, is the simplicial complex whose vertex set is P and whose faces are the chains of P.

9. Computing the Möbius Function

We will explain a few techniques for computing the Möbius function.

9.1. **The Product Formula.**

Theorem 9.1.1. *Let P and Q be posets. Then $I(P \times Q) = I(P) \otimes_{\mathbb{C}} I(Q)$.*

Proof. See [5, 348-349] for proof. □

Corollary 9.1.1 (The Product Formula)**.** *If $[(x, y), (x', y')] \in \text{Int}(P \times Q)$, then $\mu_{P\times Q}([(x, y), (x', y')]) = \mu_P([x, x'])\mu_Q([y, y'])$.*

Proof. This follows directly from Theorem 9.1.1 □

9.2. **The Reduced Euler Characteristic.**

Throughout this subsection assume P is a locally finite poset.

Definition 9.2.1. Let Δ be a simplicial complex and let f_k be the number of faces of Δ of dimension k. The *reduced Euler characteristic* is the number $\tilde{\chi}(\Delta) := \sum_{k=-1}^{\infty}(-1)^k f_k = -f_{-1} + f_0 - f_1 + f_2 - \cdots$.

Theorem 9.2.1. *Let $[x, y] \in \text{Int}(P)$ and let c_k denote the number of chains from x to y of length k. Then $\mu([x, y]) = \sum_{k=0}^{\infty}(-1)^k c_k = c_0 - c_1 + c_2 - c_3 + \cdots$.*

Proof. Notice that $\mu = \zeta^{-1} = (\delta + (\zeta - \delta))^{-1} = (\delta + \kappa)^{-1} = (\delta - (-\kappa))^{-1} = \sum_{k=0}^{\infty}(-\kappa)^k = \sum_{k=0}^{\infty}(-1)^k\kappa^k$. Given $[x, y] \in \text{Int}(P)$, it follows that $\mu([x, y]) = \sum_{k=0}^{\infty}(-1)^k\kappa^k([x, y]) = \sum_{k=0}^{\infty}(-1)^k c_k$. □

Corollary 9.2.1. *Let $[x, y] \in \text{Int}(P)$. Then $\mu_P([x, y]) = \mu_{P^*}([y, x])$.*

Proof. The statement of Theorem 9.2.1 is self-dual. □

Theorem 9.2.2. *Let $[x, y] \in \text{Int}(P)$ s.t. (x, y) is not empty. Then $\mu([x, y]) = \tilde{\chi}(\Delta((x, y)))$.*

Proof. Assume $[x, y] \in \text{Int}(P)$ s.t. $(x, y) \neq \emptyset$. Let c_k be the number of chains from x to y of length k and f_k be the number of faces of $\Delta((x, y))$ of dimension k. Notice then that $c_0 = 0$, $c_1 = 1$, and $f_{-1} = 1$ (f_{-1} counts the empty set). Given $k \geq 2$, $c_k = f_{k-2}$ since a chain $x = x_0 < x_1 < \cdots < x_k = y$ of length k contains the face $x_1 < x_2 < \cdots < x_{k-1}$ of dimension $k - 2$. Therefore, $\sum_{k=0}^{\infty} c_k = \sum_{k=-1}^{\infty} f_k$; i.e., $\mu([x, y]) = \tilde{\chi}(\Delta((x, y)))$. □

9.3. **Homological Interpretations.**

Recall that for any given simplicial complex Δ, one associates a topological space with Δ, called the *geometric realization* of Δ, denoted $|\Delta|$. The reduced Euler characteristic is classically defined by the formula $\tilde{\chi}(|\Delta|) = \sum_p(-1)^p\text{rank}(\tilde{H}_p(|\Delta|, \mathbb{Z}))$. By definition, $\tilde{\chi}(\Delta) = \tilde{\chi}(|\Delta|)$. Therefore, if (x, y) is not empty, then $\mu([x, y])$ depends only on the geometric realization $|\Delta((x, y))|$ of $\Delta((x, y))$.

10. Other Enumerative Techniques

In this section we explain other enumerative techniques and tools built on the theory of the previous sections.

10.1. **Zeta Polynomial.** Throughout this subsection, let P be a finite poset.

Definition 10.1.1. A *multichain of length* n of P is a sequence $x_1 \leq x_2 \leq \cdots \leq x_n$ of elements of P.

Definition 10.1.2. For $n \geq 2$, define $Z(P, n)$ to be the number of multichains of length $n-1$ of P. Regarded as a function of n, $Z(P, n)$ is the *zeta polynomial* of P.

Lemma 10.1.1. $Z(P, 2) = |P|$.

Proof. This is obvious from the definition of $Z(P, n)$, since $Z(P, 2)$ counts the number of multichains of length 1; i.e., the number of elements of P. □

Lemma 10.1.2. *For each $i \geq 2$, let b_i be the number of chains of P of length $i-1$. Then $Z(P, n) = \sum_{i\geq 2} b_i \binom{n-2}{i-2}$.*

Proof. The number of multichains of length $n-1$ having a chain of length $i-1$ as support is equal to $\left(\binom{i-1}{n-1-(i-1)}\right)$. □

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Berton A. Earnshaw, College of Engineering and Technology, Brigham Young University, Provo, UT 84602 U.S.A.

E-mail address: berton@et.byu.edu