ON THE NONCROSSING PARTITIONS OF A CYCLE

G. KREWERAS

Abstract. This article defines the paritions of a finite set structured in a cycle which possesses the property that a pair of points belonging to a class and a pair of points belonging to another class cannot be in a crossed way. It establishes that these partitions form a lattice and it specifies some of the descriptive and enumerative properties of the lattice; it computes in particular the Möbius function.

1. Definitions

In all that follows, we call *cycle* the pair (M, c) formed by

(1) a nonempty, finite set M of cardinality m,
(2) a circular bijection c of M into itself, where the word *circular* means that for all $x \in M$ and for all $i \in \{1, 2, \ldots, m-1\}$ we have $c^i(x) \neq x$. The elements of M are called *points*.

Let A be any nonempty subset of M, and let $x \in A$. If k_x is the least positive integer such that $c^{k_x}(x) \in A$, we put $c^{k_x}(x) = d(x)$. It is clear that $d(x)$ defines a circular bijection of A into itself; (A, d) is thus a cycle, and we call it the *trace* of (M, c) over A.

For all pairs (x, y) of distinct points of M, we call $\delta(x, y)$ (*distance* from x to y) the least positive integer k such that $c^k(x) = y$; thus we have, for all pairs $\{x, y\}$, $\delta(x, y) + \delta(y, x) = m$.

Given two disjoint pairs $\{x, y\}$ and $\{u, v\}$, we say that the pairs are *crossed* if the integer $\delta(x, y)$ is between the lesser and greater of the the two integers $\delta(x, u)$ and $\delta(x, v)$, else the two are *uncrossed*.

Any two disjoint subsets A and B of M are said to be *noncrossing* if there does not exist two crossed pairs contained in A and B respectively; in particular if at least one of the two disjoint subsets A and B is a singleton (subset of cardinality 1), A and B are necessarily noncrossing.

In certain cases we will consider two noncrossing subsets A and B of M which possess the following property: there exists two points x and y such that

$$x \in A, \ y \in B, \ c(x) \in B, \ c(y) \in A.$$

If this is so, we say that the two subsets A and B are *adjacent*. We note that one of the two adjacent subsets can be a singleton $\{x\}$; the other then contains $c(x)$ and $c^{-1}(x)$.

Given a cycle (M, c), we call *noncrossing partition* of M a partition in which any two distinct classes are noncrossing.

Date: March 7, 2005.
Translated by Berton A. Earnshaw.

The central object of this article is to study the properties of the set of noncrossing partitions.

2. Lattice Structure

Given a cycle (M, c) and any partition P of M, we define a new partition $\overline{P}$, called the *noncrossing closure* of P: the classes of P we will take as vertices of a nonoriented graph $G(P)$, two vertices of $G(P)$ are nonadjacent if and only if the two corresponding classes of P are noncrossing. It is then the vertices of each of the connected components of $G(P)$ which define the classes of P joined together to form each of the classes of $\overline{P}$. (In other words, if two classes are crossed, they join into one, and so on until there is nothing but noncrossing classes.) Every partition P of M is evidently more fine (meaning larger[1]) then its noncrossing closure.

Theorem 1. *Given any partition P of M, every noncrossing partition less fine than P is also less fine than the noncrossing closure of P.*

Proof. If Q is a partition less fine than P, every class A of P is contained in a class B of Q; let B_1 and B_2 be two distinct classes of Q and let A_1 and A_2 be two classes of P such that

$$A_1 \subseteq B_1, \; A_2 \subseteq B_2.$$

If the partition Q is noncrossing, B_1 and B_2 are noncrossing, so also A_1 and A_2. It follows that each time two classes of P are crossing, they are contained in the same class of Q. Gradually, one sees that each of the connected components of $G(P)$ whose vertices are the classes of P is contained in the same class of Q. Thus every class of the noncrossing closure of P is contained in a class of Q, proving Theorem 1. □

It is well-known that the the set $\{P, Q, \ldots\}$ of all partitions of a given set form a lattice, of which we note the two operations $P \frown Q$ (the least fine of the partitions more fine than P and Q) and $P \smile Q$ (the most fine of the partitions less fine than P and Q).

Theorem 2. *If P and Q are two noncrossing partitions, the same is true of $P \frown Q$.*

Proof. Every class of $P \frown Q$ is an intersection of a class of P and a class of Q. Now any two classes of P are by hypothesis noncrossing; it is thus the same with their repective intersections with any class of Q, and, for a stronger reason, with two distinct classes of Q. □

Theorem 3. *If P and Q are two noncrossing partitions, every noncrossing partition less fine than P and Q is less fine than the noncrossing closure of $P \smile Q$.*

Proof. Every partition less fine than P and Q is less fine than $P \smile Q$ (by definition of $P \smile Q$). If one such partition is also noncrossing, it is less fine than the noncrossing closure of $P \smile Q$ by Theorem 1. □

It follows from Theorems 2 and 3 that the noncrossing partitions of M form a set T_m that has the structure of a lattice. It is however to be noted that T_m is not in general a sublattice of the lattice of all the partitions of M. Figure 1 represents the lattice T_4 of M formed from the four points a, b, c, d placed in their cyclic order.

[1] We always use, in everything that follows, the expressions "less fine" and "more fine" in the large sense, unless otherwise indicated.

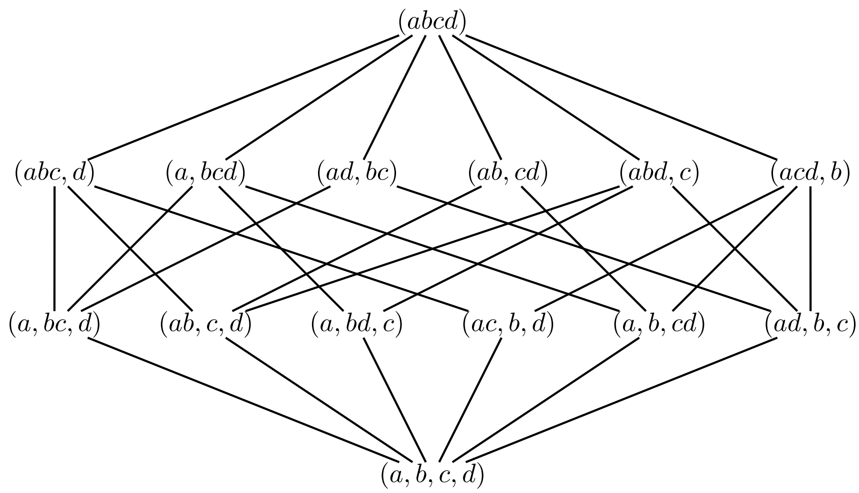

FIGURE 1

3. Dense Partitions, Complementary Partitions

In view of the study of certain properties of T_m it is interesting to consider a particular species of noncrossing partitions. Let (L, e) be a cycle where L is a set of cardinality $2m$ and e a circular bijection of L into L; we consider L as partitioned into two classes of cardinality m, one formed from the points $x, e^2(x), e^4(x), \ldots$ (even points), and the other from the points $e(x), e^3(x), \ldots$ (odd points).

This said, we call *dense partition* of L any partition R satisfying the following conditions:

(1) R is a noncrossing partition,
(2) each class of R is formed of points of the same parity,
(3) for any x, the two classes containing x and $e(x)$ respectively are adjacent (as defined in Section 1).

Every class A of a dense partition R has one or more adjacent classes; in fact, it is easy to see that there are as many as the points of A. One also sees without difficulty that if one advances beginning with A, from class to class by successive adjacencies, one can reach any class of R and one can never (without turning back) return to A.[2] It follows that the classes of R, together with its adjacencies, define a *tree*. Now the number of edges of this tree, that is to say the number of pairs of adjacent classes, is equal to m; indeed, each of the $2m$ pairs $\{x, e(x)\}$ occurs in adjacent classes and each adjacency creates (by definition) two such pairs. The trees of classes, having m edges, have thus $m + 1$ vertices. *Every dense partition of L is thus a partition into $m + 1$ classes*; Figure 2 gives an example corresponding to $m = 8$.[3]

[2]trans.: I am not sure how to translate this sentence. The translation given seems to be in accordance with what Kreweras wrote, but is obviously contrary the truth. I think the idea is that, beginning with any edge (adjacency), we do not cross the same edge until we have first walked *all* the edges of the described graph.

[3]trans.: Those familiar with meanders will recognize dense partitions as a particular collection of closed planar meanders of order m.

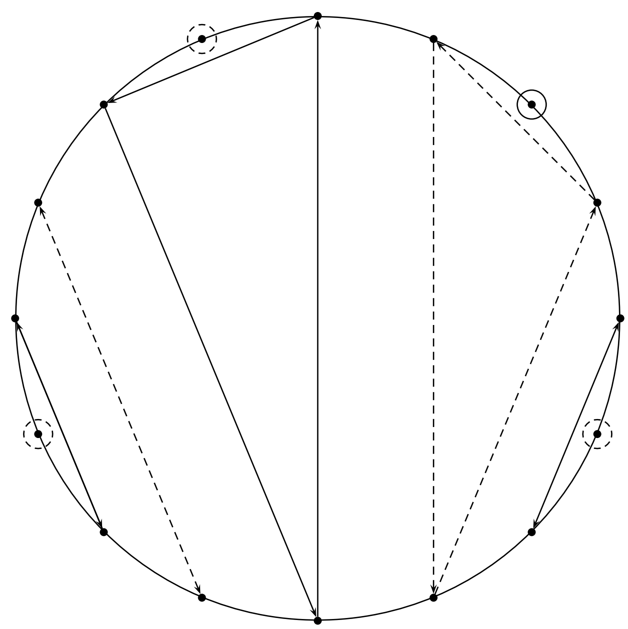

FIGURE 2

From this remark relative to the dense partitions of L results a property of the set of noncrossing partitions of M. Indeed let P be a noncrossing partition of M into h nonempty classes. The cycle (M, c) can always be considered as the trace over M of a cycle (L, e), with $e^2 = c$, which amounts to inserting between the m points of M, considered as even points, as many odd points to form another cycle (M', c') isomorphic to (M, c).[4]

Now from the partition P of M, one can always complete, by adjacent classes, a dense partition of R of L, of which the restriction to M' will be a noncrossing partition P' of M'. The latter will have $m + 1 - h$ classes since there is a total of $m+1$ classes in R. One thus sees that there will be as many noncrossing partitions of M into h classes as noncrossing partitions of M (or of M') into $m-h+1$ classes; we will later calculate the exact number of these partitions.

We now indicate a translation of this property into algebraic language. Given a cycle (M, c) and a noncrossing partition P of M, one can to this partition associate a bijection p of M into M defined as follows: every point x of M will have as its image $p(x)$ the first $c^i(x)$ $(i > 0)$ which belongs to the same class as x. Under these conditions, one can assure without difficulty that the partition P' defined earlier is isomorphic to a noncrossing partition of M (and also of M', where the x's are isomorphic to their corresponding $e(x)$'s), and that this partition P' can be defined

[4]trans.: Notice that this is nothing more than a way to describe the self-duality of T_m. The translator thinks this a very clever approach.

by

$$p' = cp^{-1}$$

as the partition P has been by p. Upon repeating the operation, one has

$$p'' = cp'^{-1} = c(pc^{-1}) = cpc^{-1}.$$

One then ends up not at the partition P from which one began, but at the partition that is constructed if one transforms every point by c.[5]

Finally another consequence of the remark relative to dense partitions is the following: if one calls *complementary* the partitions P of M and P' of M', every singleton $\{x\}$ of one of the two partitions, say P, is adjacent to the class of P' which contains the two points $e(x)$ and $e^{-1}(x)$. Now these latter points are consecutive in M' since $e = e^2 \cdot e^{-1} = c \cdot e^{-1}$. Conversely, every class of P' that contains two consecutive points y and $c(y)$ is adjacent to the singleton $\{e(y)\}$. One notably concludes that if the partition P does not contain a singleton, no class of P' contains two consecutive points (we say that P' is a *diluted* partition). It follows that, within the noncrossing partitions, the number of diluted partitions of M into α classes is equal to those partitions without a singleton of M' (or of M) into $m-\alpha+1$ classes. We will later make this number precise as a function of m and α.

4. Partitions of a Given Type

We say that a partition P is of a given *type* if one specifies, for every positive integer k, the number s_k of classes of P having cardinality k. We denote the type by the integer sequence

$$Y = ((s_1, s_2, \ldots, s_k, \ldots)).$$

If P has h nonempty class total, one clearly has

$$s_1 + s_2 + \ldots + s_k + \ldots = h,$$

$$s_1 + 2s_2 + \ldots + ks_k + \ldots = m.$$

Another way of specifying the type Y is to write the Young sequence

$$Y = (y_1, y_2, \ldots, y_h),$$

which enumerates the cardinalities of the h classes in a non-increasing order.

We denote by $[[m, h]]$ the set of all types of partitions of M into h (nonempty) classes.

Theorem 4. *If $Y \in [[m, h]]$, the number of noncrossing partitions of M having the type Y is equal to*

$$v(Y) = \frac{(m)_{h-1}}{s1!s2! \cdots sk! \cdots}.$$

Proof. The statement of the theorem is equivalent to affirming that if one computes not just the noncrossing partitions themselves, but the noncrossing partitions together with a labelling of the subsets of the same cardinality, then their number should be equal to $(m)_{h-1}$. This amounts to computing the different ways of specifying in M a *sequence* of h subsets $A_1, A_2, \ldots, A_h$, having fixed successive cardinalities $a_1, a_2, \ldots, a_h$, such that $\{A_1, \ldots, A_h\}$ is a noncrossing partition of M. This is how we will proceed. The beginning of this proof will be an induction on m.

[5]trans.: I am not quite sure what this sentence means. I think that the self-duality of T_m is supposed to be expressed here, but the statement seems to indicate otherwise.

Given any proper subset A of M, we call *gap* of A every maximal succession of points of $M \setminus A$, that is to say every sequence $x, c(x), \ldots, c^{k-1}(x)$ of elements not belonging to A such that $c^{-1}(x) \in A$ and $c^k(x) \in A$. We call x and $c^{k-1}(x)$ the *initial point* and *final point* of the gap; it is possible that they coincide.

If A is a class of the noncrossing partition P, every other class A' of the same partition P is included entirely in one gap of A; for if two elements u and v of A' belong to two distinct gaps, the pair $\{u, v\}$ is necessarily crossed with A.

Every gap C of the class A of P is thus a union of classes of P. These classes form moreover a noncrossing partition of C, if one admits the definition on C a cycle that is the trace over (M, c). Particularly if on considers the subset A_h of a noncrossing partition $P = \{A_1, A_2, \ldots, A_{h-1}, A_h\}$ having l gaps, each of these gaps will have as cardinailty a sum of the positive integers found among the terms of the sequence $a_1, a_2, \ldots, a_{h-1}$.

We calculate first the number of subsets A of M having l gaps, which we will call $C_1, C_2, \ldots, C_l$ of fixed respective cardinalities $c_1, c_2, \ldots, c_l$; A is then of cardinality $a = m - (c_1 + \cdots + c_2)$. Once C_1 is placed, which can be done in m ways (for example the m positions of the initial point of C_1) there exists $(l-1)!$ ways to specify the order of appearance after C_1 of the $l-1$ other gaps. It remains to specify how many points of A will be placed between each gap and the next, that is to say to define a sequence of l positive integers that sum to a; one knows that there are $\binom{a-1}{l-1}$ ways to do this. Finally, the number of ways of defining A is equal to the product

$$m(l-1)!\binom{a-1}{l-1} = m(a-1)_{l-1}.$$

Note the this number just depends, for given M, on the cardinality a of A and the number l of gaps, without making reference to the particular cardinalities of the gaps.

To compute the ways of taking from M the subsets $A_1, A_2, \ldots, A_h$, of fixed cardinalities $a_1, a_2, \ldots, a_h$, that constitute a noncrossing partition P, we first fix an arbitrary number l of gaps of A_h. For each of the $m(a_h - 1)_{l-1}$ possiblities relative to A_h, the set $\{1, 2, \ldots, h-1\}$ of indices of the other classes A_i partition themselves into l classes D_j $(j \in \{1, 2, \ldots, l\})$, of which each will correspond to all the A_i include in the same gap C_j of A_h; we denote this partition of $\{1, 2, \ldots, h-1\}$ by $R = \{D_1, D_2, \ldots, D_l\}$. The cardinality c_j of the gap C_j is equal to

$$c_j = \sum_{i \in D_j} a_i = a_{D_j}.$$

The classes A_i such that $i \in D_j$ constitute, one remarks, a noncrossing partition of C_j into d_j classes ($d_j = \operatorname{card} D_j$). As a_{D_j}, in as far as it is the cardinality of a gap, is certainly $\leq m-1$, one can use the induction hypothesis to affirm that the number of ways of placing on C_i the A_i for each $i \in D_j$ is equal to $(a_{D_j})_{d_j-1}$. The total number of ways of placing $A_1, A_2, \ldots, A_{h-1}$, according to the given partition R, is thus

$$X_R = \prod_{j=1}^{l} (a_{D_j}) d_{j-1}.$$

If, in leaving l fixed, one associates with R the set of all the partitions of $\{1, 2, \ldots, h-1\}$ in l classes, one can calculate the sum of the X_R by using a formal identity, for the demonstration of which we refer to [3]. By this identity, the sum is equal to

$$\binom{h-2}{l-1}(a_1+a_2+\cdots+a_{h-1})_{h-l-1}=\binom{h-2}{l-1}(m-a_h)_{h-l-1}.$$

Place in M a class A_h of cardinality a_h at one of the l gaps and distribute the $h-1$ other classes into these l gaps so that they form a noncrossing partition, so that finally the possible number of ways is equal to

$$m(a_h-1)_{l-1}\binom{h-2}{l-1}(m-a_h)_{h-l-1}=m\binom{h-2}{l-1}(a_h-1)_{l-1}(m-a_h)_{h-l-1}.$$

This expression, if at last one sums the formula over l (Vandermonde binomial formula), gives

$$m(m-1)_{h-2}=(m)_{h-1};$$

this is exactly the desired expression, finishing the proof of Theorem 4. □

Corollary 1. *The total number of noncrossing partitions of a cycle of m points into h classes is equal to*

$$\frac{(m-1)!m!}{(h-1)!h!(m-h)!(m-h+1)!}=\gamma(m-1,h-1).$$

Proof. This follows from Theorem 4 and the well-known fact that

$$\sum_{Y\in[[m,h]]}\frac{h!}{s_1!s_2!\cdots}=\binom{m-1}{h-1}.$$

This formula expresses, recall, that among the $\binom{m-1}{h-1}$ sequences of h positive integers that sum to m, the number of those that for all k having s_k terms equal to k is equal to the multinomial $\binom{h}{s_1,s_2,\ldots}$.

The same expression of $\gamma(m-1,h-1)$ confirms the result obtained in Section 3, stating that there are as many noncrossing partitions of M into h classes as there are into $m-h+1$ classes. □

Corollary 2. *The total number of noncrossing partitions of a cycle of m points is equal to the number (said "of Catalan")*

$$\gamma_m=\frac{(2m)!}{m!(m+1)!}.$$

Proof. One obtains this number by summation, over $h\in\{1,2,\ldots,m\}$, of the expression $\gamma(m-1,h-1)$; the fact that this summation gives the Catalan number is easy to establish and, as a matter of fact, well-known; cf. [2] for example. □

5. Diluted Partitions and Partitions without a Singleton

For every noncrossing partitions P (other than the trivial partition) of M into h classes, we call *arc* every maximal succession of points of the same class of P, and we consider the set N, of cardinality n, of the initial points of these arcs. It is clear that the trace of P over N is a *diluted* partition of N into h classes (the cycle on N being the trace of the cycle on M). We call then $\omega(n,h)$ the number of diluted partitions of a cycle of n elements into h classes.

Since these are $\binom{m}{n}$ ways to specify on M the n points that form N, the total number of noncrossing partitions of M into h classes can be written

$$\theta(m-1,h-1)=\sum_{n\geq h}\binom{m}{n}\omega(n,h),$$

which immediately gives

$$\frac{(m)_h(m-1)_{h-2}}{(h-1)!h!}=\sum_{k\geq 0}\frac{\omega(h+k,h)}{(h+k)!}(m)_{h+k};$$

of course, after simplification by $(m)_h$,

$$(m-1)_{h-2}=(h-1)!h!\sum_{k\geq 0}\frac{\omega(h+k,h)}{(h+k)!}(m-h)_k.$$

But one such expression of $(m-1)_{h-2}$ as a linear combination of terms $(m-h)_k$ is necessarily identical to that given by the Vandermonde formula:

$$(m-1)_{h-2}=\sum_{k\geq 0}\binom{h-2}{k}(h-1)_{h-k-2}(m-h)_k.$$

One immediately concludes that

$$\omega(h+k,h)=\frac{(h+k)!}{h(h-1)k!(k+1)!(h-k-2)!};$$

whence it is easy to get $\omega(n,h)$, of which Table 1 gives the first values. The number $\omega(m,\alpha)$ responds to the question left unanswered in Section 3 on the number of diluted partitions of M into α classes; the same number, which can be written $\omega(m,m-\beta+1)$, counts the number of partitions without singletons of M into $\beta=m-\alpha+1$ classes.

$\omega(n,h)$	$n=2$	3	4	5	6	7	8	9	10	11	12
$h=2$	1										
3		1	2								
4			1	5	5						
5				1	9	21	14				
6					1	14	56	84	42		
7						1	20	120	300	330	132

TABLE 1

It is to be noted that the numbers of Table 1 are those which occur as coefficients (or more correctly sums of the coefficients of the terms of the same "weight") of the expressions giving the b's as a function of the a's when

$$y=x(1-a_1x-a_2x^2-\cdots-a_nx^n-\cdots),$$
$$x=y(1+b_1x+b_2x^2+\cdots+b_nx^n+\cdots);$$

there is more on this subject in [1], [5], and [6].

In addition, the sums of the $\omega(n,h)$ with respect to n, by successive h's, are the numbers that solve the problem sometimes called "Schröder's parenenthisizing" (cf. [8]). One can find these numbers thanks to the following remark: all partitions without singletons of M into β classes is of the type Y' defined by a Young sequence of β terms all ≥ 2. If one collects together all the terms Y', one obtains a Young sequence Y belonging to $[[m-\beta,\beta]]$. The desired number is thus (Theorem 4)

$$\sum v(Y') = \sum \frac{(m)_{\beta-1}}{s_1!s_2!\cdots s_k!\cdots},$$

the summation over the two members, being understood as all the Y' s.t. $Y \in [[m-\beta,\beta]]$. If one takes into account that $Y = ((s_1, s_2, s_3, \ldots))$ is equivalent to $Y' = ((0, s_1, s_2, \ldots))$, one is lead to

$$\frac{(m)_{\beta-1}}{\beta!} \sum \binom{\beta}{s_1 s_2 \cdots s_k \cdots},$$

where $\sum \binom{\beta}{s_1 s_2 \cdots s_k \cdots}$ is the total number of β-compositions of the integer $m-\beta$, being $\binom{m-\beta-1}{\beta-1}$, which makes appear the desired expression.

6. Monotonic Sequences and Chains in T_m

Theorem 5. *The number of ways of which one can define in T_m a sequence of $r-1$ partitions of which each is more fine the preceeding is*

$$w(m,r) = \frac{(mr)_{m-1}}{m!}.$$

Proof. $w(m,1)$ is equal to 1 by natural convention and $w(m,2)$ reduces to the cardinality of T_m, which, as one has seen, is the Catalan number

$$\frac{(2m)!}{m!(m+1)!} = \frac{(2m)_{m-1}}{m!}.$$

The value of $w(m,r)$ in the statement of the theorem is thus valid for $r=1$ and for $r=2$; we will show that if the statement is established just for the value r of the second argument, then it is also true for the value $r+1$. We thus note that if one calls $P_1, P_2, \ldots, P_r$ a sequence of r partitions such that P_i is more fine than P_{i-1} ($i \in \{2, 3, \ldots, r\}$), and if P_i is a given partition P of type $((s_1, s_2, \ldots, s_k, \ldots)) \in [[m,h]]$, the number of ways of specifying the rest of the sequenceis the product Π of the numbers $w(k_A, r)$ for the different classes A of P_i.

Suppose that the initial given partition P is of type $Y = ((s_1, s_2, \ldots, s_k, \ldots)) \in [[m,h]]$. The product of the $w(k_A, r)$ is thus equal to

$$\Pi_\gamma = [w(1,r)]^{s_1} [w(2,r)]^{s_2} \cdots [w(k,r)]^{s_k} \cdots .$$

By the induction hypothesis, one can write

$$\Pi_\gamma = \frac{[(r)_0]^{s_1} [(2r)_1]^{s_2} \cdots [(kr)_{k-1}]^{s_k} \cdots}{(1!)^{s_1} (2!)^{s_2} \cdots (k!)^{s_k} \cdots}.$$

If instead of specifying P one specifies only the type of P, the product Π_γ is to be taken as many times as exists *noncrossing* partitions of the type Y, that is to say, by Theorem 4,

$$\frac{(m_{h-1})}{s_1!s_2!\cdots s_k!\cdots}$$

times. The number of possibilities is thus equal to

$$\frac{(m)_{h-1}[(r)_0]^{s_1} [(2r)_1]^{s_2} \cdots [(kr)_{k-1}]^{s_k} \cdots}{(1!)^{s_1} s_1! (2!)^{s_2} s_2! \cdots (k!)^{s_k} s_k! \cdots} =$$

$$\frac{\varphi(Y)}{(m-h+1)!}[(r)_0]^{s_1}[(2r)_1]^{s_2}\cdots[(kr)_{k-1}]^{s_k}\cdots;$$

$\varphi(Y)$ is the well-known expression of the total number of partitions of type Y of a given finite set.

Finally if, instead of giving the type Y of $[[m,h]]$, one associates with Y this set $[[m,h]]$, it will be calculate, as the numerator of a fraction of denominator $(m-h+1)!$, the sum

$$\sum_{Y\in[[m,h]]}\varphi(Y)\frac{[(r)_0]^{s_1}[(2r)_1]^{s_2}\cdots[(kr)_{k-1}]^{s_k}\cdots}{}.$$

But since $\varphi(Y)$ is the total number of partitions of type Y of a set of cardinality m, the above sum appears as a particular case of the first member of the identity already used in Section 4, which is the case where the m variables x_i are all equal to r: the sum X_A for a class A of cardinality k is equal to kr, and the particular subset is equal to the product which was noted X_P in the particular identity.

The desired sum is none other than the sum of these X_P extended to all the partitions P into h nonempty classes, and the identity tells us the value of the sum; after reintroducing the denominator $(m-h+1)!$, one has thus the total number of possibilities equal to

$$\frac{\binom{m-1}{h-1}(mr)_{m-h}}{(m-h+1)!}=\frac{1}{mr+1}\binom{m-1}{h-1}C_{mr+1}^{m-(h-1)}.$$

This is thus the number of sequences $P_1, P_2, \ldots, P_r$ desired if on fixes the initial noncrossing partition P_1 composed of h nonempty classes. If suffices to sum this last expression for h between 1 and m, which is done without difficulty, to obtain the final number of possibilities

$$w(m,r+1)=\frac{[m(r+1)]_{m-1}}{m!};$$

the theorem is thus established. □

Corollary 3. *The number of sequences of $r-1$ noncrossing partitions of which each is strictly more fine than the preceeding, the first having at least two classes and the last having at most $m-1$, is equal to*

$$\frac{(rm)_{m-1}}{m!}-\binom{r}{1}\frac{[(r-1)m]_{m-1}}{m!}+\binom{r}{2}\frac{[(r-2)m]_{m-1}}{m!}-\cdots+(-1)^{r-1}\binom{r}{r-1},$$

the rth difference, evaluated at $x=0$, of the polynomial $\frac{(mx)_{m-1}}{m!}$.

Proof. This corollary is immediately established with the help of the principle of inclusion-exclusion. □

Corollary 4. *The number of chains joining in T_m the trivial partition (into* 1 *class) to the discrete partition (into m classes) is m^{m-2}.*

Proof. This results from Corrolary 3 applies to the case where $r=m-1$. Under the polynomial $(mx)_{m-1}$, which is of degree $m-1$ in x, having only one term of degree $m-1$ in x, which is equal to $m^{m-1}x^{m-1}$, having a nonzero $(m-1)$th difference: this difference is equal to $m^{m-1}(m-1)!$, which after division by $m!$ gives exactly m^{m-2}. □

The same result has been established by Poupard [4] by putting these chains into one-to-one correspondence with the set of trees having m given vertices.

7. Möbius Function

Theorem 6. *The Möbius function*[6] *of* T_m *between* 0 *(discrete partition) and* M *(trivial partition), is equal to*

$$\mu(0,M) = (-1)^{m-1}\frac{(2m-2)!}{(m-1)!m!} = \frac{(-m)_{m-1}}{m!} = \theta_m.$$

Proof. This proof will proceed by induction on m.

Every element P of T_m defines a sublattice C_P of the partitions more fine than P.

If $P = \{A_1, A_2, \ldots, A_h\}$, with $h \le m-1$, and if the cardinalities of the classes correspond to $a_1, a_2, \ldots, a_h$, C_P is isomorphic to the product of the lattices $T_{a_1} \times T_{a_2} \times \cdots \times T_{a_h}$; consequently the Möbius function of T_m between 0 and P is equal to the product $\theta_{a_1}\theta_{a_2}\cdots\theta_{a_h}$.

If P is of type $Y = ((s_1, s_2, \ldots, s_k, \ldots)) \in [[m.h]]$, this product is equal to

$$\mu(0,P) = \theta_1^{s_1}\theta_2^{s_2}\cdots\theta_k^{s_k}\cdots = \frac{[(-1)_0]^{s_1}[(-2)_1]^{s_2}\cdots[(-k)_{k-1}]^{s_k}\cdots}{(1!)^{s_1}(2!)^{s_2}\cdots(k!)^{s_k}\cdots}.$$

Now the partitions of the same type Y are, by Theorem 4, in total number

$$\frac{m!}{(m-h+1)!}\frac{1}{s_1!s_2!\cdots s_k!\cdots}.$$

The sum of the corresponding values of $\mu(0,P)$ is thus

$$\frac{\varphi(Y)}{(m-h+1)!}[(-1)_0]^{s_1}[(-2)_1]^{s_2}\cdots[(-k)_{k-1}]^{s_k}\cdots,$$

an expression encountered in the statement of Theorem 5 and with the same meaning of $\varphi(Y)$, but replacing r by -1.

If we do not specify the type Y, but instead associate to Y the set $[[m,h]]$, the calculation of the sum of the corresponding $\mu(0,P)$ follows as before, but with all the variables x_i equal to -1. One thus has the expression

$$\frac{\binom{m-1}{h-1}(-m)_{m-h}}{(m-h+1)!}.$$

To get $\mu(0,M)$, one just sums the above expression for h between 2 and m, then changing the sign of the sum. To be assured that one gets thus $\theta_m = \frac{(-m)_{m-1}}{m!}$ that is none other than the same expression for $h=1$, it suffices to show that

$$\sum_{h=1}^{m}\frac{\binom{m-1}{h-1}(-m)_{m-h}}{(m-h+1)!} = 0.$$

But this follows from the fact that the first member can be put in the form

$$\frac{1}{(-m+1)m!}\sum_{k=0}^{m}\binom{m}{k}(u)_k(v)_{m-k},$$

with $u = m-1$ and $v = -m+1$, which gives exactly 0 by Vandermonde's binomial formula. Theorem 6 is thus proved. □

[6] Cf. Rota [7]

References

[1] Arthur Cayley. On the partitions of a polygon. *Phil. Mag.*, 4,22:237–262, 1890-1891.

[2] Germain Kreweras. Sur les éventails de segments. *Cahiers B.U.R.O.*, 15:16–22, 1970.

[3] Germain Kreweras. Une famille d'identités mettant en jeu toutes les partitions d'un ensemble fini de variables en un nombre donné de classes. *C. R. Acad. Sci. Paris Sér. A-B*, 270:A1140–A1143, 1970.

[4] Yves Poupard. Codage et dénombrement de diverses structures apparentées à celle d'arbre. *Cahiers B.U.R.O.*, 16:71–80, 1970.

[5] George N. Raney. Functional composition patterns and power series reversion. *Trans. Amer. Math. Soc.*, 94:441–451, 1960.

[6] John Riordan. *Combinatorial identities.* John Wiley & Sons Inc., New York, 1968.

[7] Gian-Carlo Rota. On the foundations of combinatorial theory. I. Theory of Möbius functions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 2:340–368 (1964), 1964.

[8] E. Schröder. Vier kombinatorische Probleme. *Z. Math. Phys.*, 15:361–376, 1870.

G. Kreweras, Institut de Statistique des Universités de Paris, 9 Quai Saint-Bernard, 75-Paris-5éme, France

Berton A. Earnshaw, College of Engineering and Technology, Brigham Young University, Provo, UT 84602 U.S.A.

E-mail address: berton@et.byu.edu