EXTERIOR BLOCKS OF NONCROSSING PARTITIONS

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ABSTRACT. This paper defines an *exterior block* of a noncrossing partition, then gives a formula for the number of noncrossing partitions of the set $[n]$ with k exterior blocks. Certain identities involving Catalan numbers are derived from this formula.

1. INTRODUCTION

A *noncrossing partitions* is a partition π of the set $[n] := \{1, 2, \ldots, n\}$ such that whenever $1 \leq a < b < c < d \leq n$ and a and c are in the same block of π and b and d are in the same block of π, then actually a, b, c and d are all in the same block of π. The collection of noncrossing partitions of $[n]$ is denoted by NC_n. We typically write noncrossing partitions using a '/' to delimit the blocks of the partition and a ',' to delimit the elements within each block. For example, the partition $\pi = \{\{1, 4, 6\}, \{2, 3\}, \{5\}, \{7\}, \{8, 10\}, \{9\}, \{11, 12\}\} \in \mathrm{NC}_{12}$ is typically written $\pi = 1, 4, 6/2, 3/5/7/8, 10/9/11, 12$. Notice that we have written the blocks in ascending order of their least element. Noncrossing partitions can be conveniently visualized in their *linear* representations; that is, we place n nodes $1, 2, \ldots, n$ in ascending order on a line, and indicate that two elements are in the same block by drawing an arc connecting the two. All the arcs must be drawn in the same half-plane. Figure 1 gives the linear representation of $1, 4, 6/2, 3/5/7/8, 10/9/11, 12$. We will make use of the linear representation of a noncrossing partition throughout this paper.

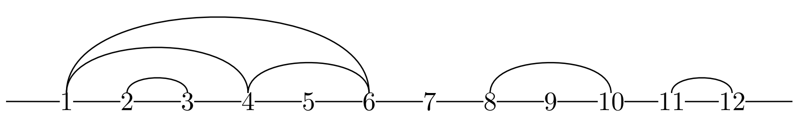

FIGURE 1. Linear representation of $1, 4, 6/2, 3/5/7/8, 10/9/11, 12$

2. EXTERIOR BLOCKS AND THE FUNCTION $\mathrm{e}(n, k)$

For ease of discussion we give a preliminary defintion. Given a block $B \in \pi$, we will denote the least and greatest elements of B by $\mathrm{first}(B)$ and $\mathrm{last}(B)$, respectively, and will call them the *first* and *last* elements of B, respectively.

Key words and phrases. noncrossing partition, exterior block, Catalan triangle, Catalan identity.

Definition 2.1. Let $\pi \in \mathrm{NC}_n$. A block $B \in \pi$ is an *interior block* of π if there exists a block $C \in \pi$ such that $\mathrm{first}(C) < \mathrm{first}(B) \leq \mathrm{last}(B) < \mathrm{last}(C)$. If B is not an interior block, then it is an *exterior block* of π.

Intuitively, given a noncrossing partition π of $[n]$, an interior block of π is one which is nested inside another block in the linear representation of π. An exterior block of π is one which is not nested in any other block. Consider Figure 1 which is the linear representation of $\pi = 1,4,6/2,3/5/7/8,10/9/11,12 \in \mathrm{NC}_{12}$. It is easy to see that $\{2,3\},\{5\}$ and $\{9\}$ are the interior blocks of π, while $\{1,4,6\},\{7\},\{8.10\}$ and $\{11,12\}$ are the exterior blocks of π.

Let $\mathrm{E}_{n,k}$ be the subset of NC_n consisting of all noncrossing partitions of $[n]$ with k exterior blocks and define

$$\mathrm{e}(n,k) = |\mathrm{E}_{n,k}|$$

so that $\mathrm{e}(n,k)$ counts the number of noncrossing partitions of $[n]$ with k exterior blocks. What sort of function is $\mathrm{e}(n,k)$?

Propostion 2.1. $\mathrm{e}(n,k) = 0$ *whenever* $k = 0$ *or* $k > n$.

Proof. If $k = 0$, we are asking how many noncrossing partitions of $[n]$ have no exterior blocks. It is easy to see that the block containing 1 is an exterior block of any noncrossing partition. Thus $\mathrm{e}(n,0) = 0$. Since any partition of $[n]$ can have at most n blocks, it can have at most n exterior blocks. So if $k > n$, $\mathrm{e}(n,k) = 0$. □

Theorem 2.1. $\mathrm{e}(n,1) = C_{n-1}$, *where* $C_n = \frac{1}{n+1}\binom{2n}{n}$ *is the nth* Catalan number.

Proof. It is easy to see that $\mathrm{e}(1,1) = 1 = C_0$. Assume $n > 1$. It is also easy to see that a noncrossing partition π of $[n]$ with one exterior block necessarily has 1 and n in the same block. Call this block B (see Figure 2, where $n = 6$, $\pi = 1,4,6/2,3/5$ and $B = \{1,4,6\}$). The partition

$$\pi' = (\pi \setminus \{B\}) \cup \{B \setminus \{n\}\}$$

is then a noncrossing partition of $[n-1]$ (π' is simply π with the element n removed; see Figure 3). Define a map $\phi : \mathrm{E}_{n,1} \to \mathrm{NC}_{n-1}$ by the above operation $\pi \mapsto \pi'$. The map ϕ is clearly invertible, with inverse map ϕ^{-1} given by

$$\phi^{-1}(\sigma) = (\sigma \setminus \{A\}) \cup \{A \cup \{n\}\}$$

where $\sigma \in \mathrm{NC}_{n-1}$ and A is the block of σ containing the element 1 (see Figures 4 and 5, where $n = 6$, $\sigma = 1,2/3/4,5$ and $A = \{1,2\}$). Therefore ϕ is a bijection, proving

$$\mathrm{e}(n,1) = |\mathrm{E}_{n,1}| = |\mathrm{NC}_{n-1}| = C_{n-1}.$$

□

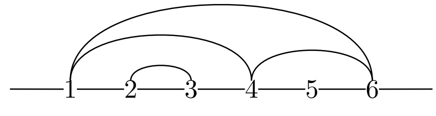

FIGURE 2. Linear representation of $\pi = 1,4,6/2,3/5 \in \mathrm{E}_{6,1}$

Theorem 2.2. $\mathrm{e}(n,k) = \mathrm{e}(n-1,k-1) + \mathrm{e}(n,k+1)$ *for* $n \geq 2$ *and* $k \geq 1$.

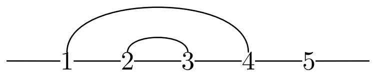

FIGURE 3. Linear representation of $\pi' = \phi(\pi) = 1,4/2,3/5 \in \mathrm{E}_{5,2}$

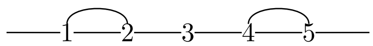

FIGURE 4. Linear representation of $\sigma = 1,2/3/4,5 \in \mathrm{E}_{5,3}$

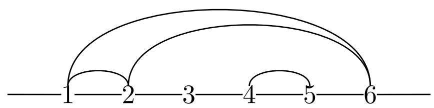

FIGURE 5. Linear representation of $\phi^{-1}(\sigma) = 1,2,6/3/4,5 \in \mathrm{E}_{6,1}$

Proof. Clearly $\mathrm{e}(n-1,k-1)$ counts the number of noncrossing partitions π of $\mathrm{E}_{n,k}$ having the singleton $\{n\}$ as a block since $\pi \setminus \{n\} \in \mathrm{E}_{n-1,k-1}$. Thus we want to show that $\mathrm{e}(n,k+1)$ counts the number of noncrossing partitions of $\mathrm{E}_{n,k}$ that do not have $\{n\}$ as a block. Let $\mathrm{E}'_{n,k}$ be that set.

It is easy to see that if $k \in [n-1]$ then there exists a noncrossing partition with k exterior blocks whose block containing n is not a singleton. Thus if $\mathrm{E}'_{n,k}$ is empty, then necessarily $k \geq n$. But then $\mathrm{e}(n,k+1) = 0$ by Propostion 2.1 and we are done.

If $\mathrm{E}'_{n,k}$ is not empty, then for any $\pi \in \mathrm{E}'_{n,k}$, let B be the block of π containing n and let

$$\pi' = (\pi \setminus \{B\}) \cup \{B \setminus \{n\}, \{n\}\}$$

(see Figures 6 and 7, where $n = 6$, $k = 2$, $\pi = 1,2/3,4,6/5$ and $B = \{3,4,6\}$). Now π' is a noncrossing partition of $[n]$ with more than k exterior blocks. Let C be the block of π' just to the right of $B \setminus \{n\}$ in the linear representation of π'; that is, $\mathrm{last}(B \setminus \{n\}) + 1 = \mathrm{first}(C)$ ($B \setminus \{6\} = \{3,4\}$ and $C = \{5\}$ in Figure 7). Let

$$\pi'' = (\pi \setminus \{C, \{n\}\}) \cup \{C \cup \{n\}\}$$

(see Figure 8). Now $\pi'' \in \mathrm{E}_{n,k+1}$. Define a map $\psi : \mathrm{E}'_{n,k} \to \mathrm{E}_{n,k+1}$ by the above operation $\pi \mapsto \pi''$. The map ψ is clearly invertible with inverse map ψ^{-1} given by

$$\psi^{-1}(\sigma) = (\sigma \setminus \{A\}) \cup \{D \cup \{n\}, A \setminus \{n\}\}$$

where $\sigma \in \mathrm{E}_{n,k+1}$ and A is the block of σ containing n and D is the block of σ just to the left of A in the linear representation of σ; that is, $\mathrm{last}(D) + 1 = \mathrm{first}(A)$ (see Figures 9 and 10, where $n = 6$, $k = 2$, $\sigma = 1,2/3/4,5,6$, $A = \{4,5,6\}$ and $D = \{3\}$). Therefore, ψ is a bijection and

$$\left|\mathrm{E}'_{n,k}\right| = |\mathrm{E}_{n,k+1}| = \mathrm{e}(n,k+1).$$

We have proven the desired recurrence. □

This recurrence relations allows us to write out a table of values for $\mathrm{e}(n,k)$ (see Figure 11). Notice that the values of the first two columns of this table come

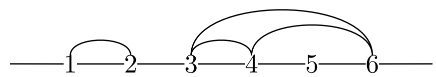

FIGURE 6. Linear representation of $\pi = 1, 2/3, 4, 6/5 \in \mathrm{E}'_{6,2}$

FIGURE 7. Linear representation of $\pi' = 1, 2/3, 4/5/6 \in \mathrm{E}_{6,4}$

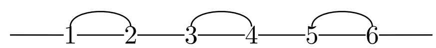

FIGURE 8. Linear representation of $\pi'' = \psi(\pi) = 1, 2/3, 4/5, 6 \in \mathrm{E}_{6,3}$

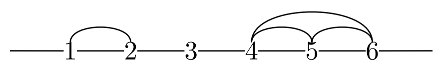

FIGURE 9. Linear representation of $\sigma = 1, 2/3/4, 5, 6 \in \mathrm{E}_{6,3}$

from Proposition 2.1 and Theorem 2.1, while the rest of the values come from the recurrence relation written as $\mathrm{e}(n, k+1) = \mathrm{e}(n, k) - \mathrm{e}(n-1, k-1)$.

Catalan numbers abound in this table. Notice that the second and third columns (corresponding to $k = 1$ and $k = 2$) contain Catalan numbers. The first column is, of course, given to us by Theorem 2.1. When $k = 2$ and $n \geq 2$, the recurrence relation plus Proposition 2.1 shows us that

$$\mathrm{e}(n, 2) = \mathrm{e}(n, 1) - \mathrm{e}(n-1, 0) = C_{n-1} - 0 = C_{n-1}.$$

Notice that the nth row adds up to C_n. This is clear since the sets

$$\mathrm{E}_{n,1}, \mathrm{E}_{n,2}, \ldots, \mathrm{E}_{n,n}$$

partition NC_n. This fact gives

$$C_n = |\mathrm{NC}_n| = |\cup_{k=1}^{n} \mathrm{E}_{n,k}| = \Sigma_{k=1}^{n} |\mathrm{E}_{n,k}| = \Sigma_{k=1}^{n} \mathrm{e}(n, k). \tag{2.1}$$

Also notice the strong resemblance of this table with the various formulations of *Catalan's triangle* (cf. [1], also sequences A053121, A008315, etc. in [4]). Figure 12 is a typical Catalan triangle. It is also called a Pascal semi-triangle since if $w(n, k)$ represents the value in the nth row and kth column of this table, then for $n \geq 1$ and $k \geq 1$, $w(n, k)$ satisfies the recurrence relation

$$w(n, k) = w(n-1, k-1) + w(n-1, k+1).$$

Notice that the diagonals $w(2n, 0), w(2n-1, 1), \ldots, w(n, n)$ of this triangle are the rows $\mathrm{e}(n+1, 1), \mathrm{e}(n+1, 2), \ldots, \mathrm{e}(n+1, n+1)$ in Figure 11.

Theorem 2.3. $\mathrm{e}(n, k) = \frac{k}{n}\binom{2n-k-1}{n-1}$.

Proof. It is easy to check that this formula satisfies Proposition 2.1 and Theorems 2.1 and 2.2. $\square$

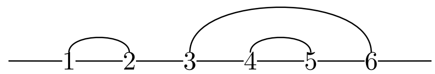

FIGURE 10. Linear representation of $\psi^{-1}(\sigma) = 1, 2/3, 6/4, 5 \in \mathrm{E}'_{6,2}$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	Total
1	0	1	0	0	0	0	0	0	0	0	0	1
2	0	1	1	0	0	0	0	0	0	0	0	2
3	0	2	2	1	0	0	0	0	0	0	0	5
4	0	5	5	3	1	0	0	0	0	0	0	14
5	0	14	14	9	4	1	0	0	0	0	0	42
6	0	42	42	28	14	5	1	0	0	0	0	132
7	0	132	132	90	48	20	6	1	0	0	0	429
8	0	429	429	297	165	75	27	7	1	0	0	1430
9	0	1430	1430	1001	572	275	110	35	8	1	0	4862
10	0	4862	4862	3432	2002	1001	429	154	44	9	1	16796

FIGURE 11. Table of values of $\mathrm{e}(n, k)$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	Total
0	1	0	0	0	0	0	0	0	0	0	0	1
1	0	1	0	0	0	0	0	0	0	0	0	1
2	1	0	1	0	0	0	0	0	0	0	0	2
3	0	2	0	1	0	0	0	0	0	0	0	3
4	2	0	3	0	1	0	0	0	0	0	0	6
5	0	5	0	4	0	1	0	0	0	0	0	10
6	5	0	9	0	5	0	1	0	0	0	0	20
7	0	14	0	14	0	6	0	1	0	0	0	35
8	14	0	28	0	20	0	7	0	1	0	0	70
9	0	42	0	48	0	27	0	8	0	1	0	126
10	42	0	90	0	75	0	35	0	9	0	1	252

FIGURE 12. A Catalan Triangle

3. CATALAN IDENTITIES

Using the formulation $\mathrm{e}(n, k) = \frac{k}{n}\binom{2n-k-1}{n-1}$ of Theorem 2.3, we can derive two identities involving Catalan numbers. The first comes by replacing $\mathrm{e}(n, k)$ in Equation 2.1 by this formula:

$$C_n = \sum_{k=1}^{n} \mathrm{e}(n, k) = \sum_{k=1}^{n} \frac{k}{n}\binom{2n-k-1}{n-1}.$$

The second identity comes by considering the number of ways the element $n+1$ can be added to a noncrossing partition π of $[n]$ to get a noncrossing partition π' of $[n+1]$. It is clear that if π has k exterior blocks, then there are $k+1$ ways to form a new noncrossing partition π': k ways by adding the element $n+1$ to each of the exterior blocks, and one way by adding the singleton $\{n+1\}$ to π. Since there are $\mathrm{e}(n,k)$ noncrossing partitions of $[n]$ with k exterior blocks, there are a total of $(k+1)\mathrm{e}(n,k)$ noncrossing partitions of $[n+1]$ gotten in this way. Summing these formulae over the possible number of exterior blocks gives

$$C_{n+1} = |\mathrm{NC}_{n+1}| = \sum_{k=1}^{n}(k+1)\mathrm{e}(n,k) = \sum_{k=1}^{n}\frac{k(k+1)}{n}\binom{2n-k-1}{n-1}.$$

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