Math 299 **Solutions to the Review for Midterm 2**

PROBLEM 1. Negate the following statement: *For all $\epsilon > 0$, there is some $\delta > 0$ such that if $|x - 3| < \delta$, then $|x^2 - 9| < \epsilon$.*

Answer: Notice that there is a hidden quantifier in the statement, which is of the form $\forall x \in$ R. The negation is: *There exists $\epsilon > 0$ such that for all $\delta > 0$, there exists some $x \in \mathbb{R}$ such that $|x - 3| < \delta$ and $|x^2 - 9| \geq \epsilon$.*

PROBLEM 2. Express the following using $\forall$ and $\exists$: *There is some $b \in \mathbb{R}$ such that the equation $x^2 - b = 0$ has no real solution.*

Answer: $\exists b \in \mathbb{R}$, *such that* $\forall x \in \mathbb{R}$, $x^2 - b \neq 0$.

PROBLEM 3. Let $n \in \mathbb{N}$.
(a) Use induction to show that exactly one element of the set $\{n, n+1, n+2, n+3\}$ is divisible by 4.

Proof: First note that there is *at most* one element which is divisible by 4, since otherwise an element of the set $\{1, 2, 3\}$ would be divisible by 4.

Now we use induction to prove that at least one element of $\{n, n+1, n+2, n+3\}$ is divisible by 4. The base case is obvious. For the inductive step, assume there is some

$$x \in \{k, k+1, k+2, k+3\}$$

that is divisible by 4. We want to show that some element in $\{k+1, k+2, k+3, k+4\}$ is divisible by 4. If $x = k+1, k+2$ or $k+3$, then we are done. If $x = k$, then $k+4$ is divisible by 4, and we are done.

(b) Use the Division Lemma to show that exactly one element of the set $\{n, n+1, n+2, n+3\}$ is divisible by 4.

Proof: It follows just as in the beginning of part (a) that there is *at most* one element of the set $\{n, n+1, n+2, n+3\}$ that is divisible by 4. We need to show there is *at least* one. Write $n = 4q + r$ for some $0 \leq r < 4$. If $r = 0$, then we are done. So we may assume $r > 0$. Let $k = 4 - r$. This is an element of $\{1, 2, 3\}$. Then $n + k = 4q + 4$ is divisible by 4, and $n + k \in \{k, k+1, k+2, k+3\}$.

PROBLEM 4. Let $x \in \mathbb{N} = \{1, 2, \ldots\}$.
(a) Prove that $x^2 + x$ is even.

Proof: If $x = 2k$ is even, then $x^2 + x = 4k^2 + 2k = 2k(2k+1)$ is even. If $x = 2k+1$ is odd, then $x^2 + x = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$ is even.

(b) Prove that $(x^2 + x)/2$ is divisible by x if and only if x is odd.

Proof 1: If $x = 2k+1$ is odd, then $(x^2 + x)/2 = 2k^2 + 3k + 1 = (2k+1)(k+1)$, which is obviously divisible by $x = 2k+1$.

For the converse use contradiction. Assume $(x^2 + x)/2$ is divisible by x, and $x = 2k$ is even. Then $(x^2 + x)/2 = k(2k+1)$. Since this is divisible by $x = 2k$, we must have that

$k(2k+1)/2k = (2k+1)/2$ is an integer. This is impossible since the numerator $2k+1$ is odd.

Proof 2: Write $(x^2+x)/2 = x(x+1)/2$. First suppose x is odd. We want to show that $x(x+1)/(2x) = (x+1)/2$ is an integer. This is immediate since the numerator $x+1$ is even.

Conversely, suppose x divides $(x^2+x)/2$. This implies that $(x+1)/2$ is an integer, and hence $x+1$ is even. It follows that x is odd.

(c) Prove that $(x^2+x)/2$ is divisible by $x+1$ if and only if x is even.

Proof: This is similar to Part (b).

PROBLEM 5. (Houston 26.7 (iii)) Show that if $x^2 - 3x + 2 < 0$, then $1 < x < 2$.

Proof: Write $x^2 - 3x + 2 = (x-1)(x-2)$. If this is negative, then we are in one of two cases:

Case 1: $x - 1 > 0$ and $x - 2 < 0$, or

Case 2: $x - 1 < 0$ and $x - 2 > 0$.

The first case is equivalent to $x > 1$ and $x < 2$, which is impossible. The second case is equivalent to $1 < x < 2$, as desired.

PROBLEM 6. (Houston 27.23 (v)) Prove that every common divisor of $a, b \in \mathbb{Z}$ is a divisor of $\gcd(a,b)$.

Proof: Suppose c divides a and b. By Theorem 28.7 we can write

$$ma + nb = \gcd(a,b)$$

for some integers $m, n \in \mathbb{Z}$. Since c divides a and b, it also divides $ma + nb$, by Theorem 27.5. It follows that c divides $\gcd(a,b)$.

PROBLEM 7. (a) Calculate $\gcd(52, 221)$.

Answer: Following the Euclidean algorithm, write

$$\begin{aligned} 221 &= 52 \times 4 + 13 \\ 52 &= 13 \times 4 + 0 \end{aligned}$$

This implies that $\gcd(52, 221) = 13$.

(b) Find $m, n \in \mathbb{Z}$ such that $52m + 221n = \gcd(52, 221)$.

Answer: Look at the top line in the answer to part (a):

$$13 = (-4) \times 52 + 1 \times 221.$$

So $m = -4$ and $n = 1$.

PROBLEM 8. Let $n \in \mathbb{N}$. Prove that n is composite if and only if n has a factor a that satisfies

$$1 < a \leq \sqrt{n}.$$

Proof: If n is composite, then we can write $n = ab$, where $a, b > 1$ are integers. Without loss of generality, we may assume $a \leq b$ (if this is not the case, then relabel a and b). Then

$$a \leq b \Rightarrow a^2 \leq ab = n.$$

Taking the square root gives

$$a \leq \sqrt{n}.$$

We already know $a > 1$, so this finishes the proof.