Math 299 Recitation 7: Existence Proofs and Mathematical Induction

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Existence proofs: To prove a statement of the form $\exists x \in S, P(x)$, we give either a constructive or a non-contructive proof. In a constructive proof, one proves the statement by exhibiting a specific $x \in S$ such that $P(x)$ is true. In a non-constructive proof, one proves the statement using an indirect proof such as a proof by contradiction. Thus, one might prove that the negation $\forall x \in S, \sim P(x)$ is false by deriving a contradiction.

Example of a constructive proof: Suppose we are to prove

$$\exists n \in \mathbb{N}, n \text{ is equal to the sum of its proper divisors.}$$

*P*roof: Let $n = 6$. The proper divisors of 6 are $1, 2$, and 3. Since $1 + 2 + 3 = 6$, we have proved the statement.

Exercise 1: Give another proof of this statement by finding a different example. (Hint: The smallest example larger than 6 happens to be a number between 25 and 30.)

An integer which is equal to the sum of its proper divisors is called a *perfect number*. An open problem is to prove or disprove the following statement: there exists an odd perfect integer.

Example of a non-constructive proof: Suppose we are to prove

$$\forall x \in \mathbb{Q}\, \exists n \in \mathbb{N}, x \leq n.$$

Proof: Suppose, by way of contradition, that there exists an $x \in \mathbb{Q}$ such that $x > n$ for every $n \in \mathbb{N}$. Since $1 \in \mathbb{N}$, we have that $x > 1$. Therefore, $x = a/b$ for some $a, b \in \mathbb{N}$ such that $a > b$. Since $a \in \mathbb{N}$, $a/b > a$. This implies that $1/b > 1$; and thus $1 > b$, which is a contradiction (since $b \in \mathbb{N}$).

Exercise 2: The statement in the previous example can be proved by giving a construction. Give a constructive proof that

$$\forall x \in \mathbb{Q}\, \exists n \in \mathbb{N}, x \leq n.$$

Intermediate Value Theorem. Suppose that $f(x)$ is a continuous function on an interval $[a, b]$. If y is a real number between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = y$.

Exercise 3: Apply the intermediate value theorem to give a non-constructive proof that for every $y \in [-1, 1]$, there exists an $x \in [-\pi/2, \pi/2]$ such that $\sin x = y$. You may assume that $f(x) = \sin x$ is continuous.

Disproving existence statements: To disprove a statement of the form $\exists x \in S, P(x)$, we prove the negation, $\forall x \in S, \sim P(x)$.

Example of disproving an existence statement: Suppose we are to disprove that there exists an $n \in \mathbb{Z}$ such that $n^2 \equiv 3 \pmod 7$.

Proof: We prove that for every $n \in \mathbb{Z}$, n^2 is not congruent to 3 modulo 7. Let $n \in \mathbb{Z}$. Then $n = 7k + r$ for some $k \in \mathbb{Z}$ and some $r \in \{0, 1, \ldots, 6\}$. Thus, $n \equiv r$. We consider the six cases:

n	$n^2 \pmod 7$
0	0
1	1
2	4
3	2
4	2
5	4
6	1

Therefore, n^2 is not congruent to 3 modulo 7.

Exercise 4 Disprove the following statement: there exists a real solution to the equation $x^4 - 2x^2 = -2$.

Principle of Mathematical Induction: Let $n \in \mathbb{N}$ and let $P(1), P(2), \cdots$ be statements. Suppose that

(1) $P(1)$ is true, and
(2) $\forall n \in \mathbb{N}$, the implication $P(n) \implies P(n+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Caution: Both (1) and (2) must hold.

Exercise 5 Find the error in the following "proof":

If $n+1 < n$, then by adding 1 to both sides, $n+2 < n+1$. Therefore, by mathematical induction, $n+1 < n$ for every $n \in \mathbb{N}$.

Exercise 6 Find the error in the following "proof":

Let $n = 1$. Then $n < 100$. Now let $n \in \mathbb{N}$. Suppose that $n - 1 < 100$. Since n is an integer, we have that $n - 1 \leq 99$. Therefore, $n \leq 100$. Hence, by mathematical induction, $\forall n \in \mathbb{N}$, $n \leq 100$.

Example of Proof by Mathematical Induction: Let $a \in \mathbb{R}$ and let $b_1, b_2, \cdots, b_n \in \mathbb{R}$. We will prove that

$$a(b_1 + b_2 + \cdots + b_n) = ab_1 + ab_2 + \cdots + ab_n.$$

Proof: Let $n = 1$. Then $a(b_1) = ab_1$ is true. Let $n = 2$. Then $a(b_1 + b_2) = ab_1 + ab_2$ by the distributive property:

$$\forall x, y, z \in \mathbb{R}, x(y + z) = xy + xz.$$

Assume that $a(b_1 + b_2 + \cdots + b_n) = ab_1 + ab_2 + \cdots ab_n$. (This is called the **inductive hypothesis**.) Then,

$$a(b_1 + b_2 + \cdots + b_{n+1}) = a(b_1 + b_2 + \cdots + b_n) + ab_{n+1}$$

by the distributive property (using $b_1 + b_2 + \cdots b_n$ in the role of y). By the inductive hypothesis,

$$a(b_1 + b_2 + \cdots + b_n) + ab_{n+1} = ab_1 + ab_2 + \cdots + ab_n + ab_{n+1}.$$

Therefore, by the principle of mathematical induction, $\forall n \in \mathbb{N}$, $a(b_1 + b_2 + \cdots + b_n) = ab_1 + ab_2 + \cdots + ab_n$.

Comments on the Proof: We verified $n = 1$ separately as this was an exceptional case. We verified $n = 2$ using the distributive property. This method also worked to prove the implication $P(n) \implies P(n+1)$. This situation is not uncommon. Sometimes a statement is not true for the first positive integers or these statements might be true but for exceptional reasons. The Principle of Mathematical Induction applies if you have a sequence of statements $P(n)$ for all n greater than or equal to some $k \in \mathbb{Z}$ (possibly a negative integer). Refer to section 6.2 for additional discussion.

Exercise 7 Let x and y be nonnegative real numbers such that $x \leq y$. Use mathematical induction to prove that

$$\forall n \in \mathbb{N}, x^n \leq y^n.$$

Exercise 8 Use induction to prove that for every positive integer n, $n! \leq n^n$.

Additional Exercises:

1. Prove that there exists and irrational number in the interval $(10^{-2014}, 10^{-2013})$.
2. Generalize the previous exercise: prove that every nonempty open interval contains an irrational number.

Math 299 **Quiz 7**

Please answer each question in the space provided. Use complete sentences and correct mathematical notation to write your answers. You have 20 minutes to complete this quiz.

1. (5 points) Use induction to prove that for each $n \in \mathbb{N}$,

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

2. (5 points) Prove or disprove the statement: "The number $1 + n + n^2$ is odd for every integer n."

3. (5 points) Let $x \in \mathbb{R}$ and assume that $x \neq 1$. Use induction to prove that for each nonnegative integer n,

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

You must use induction to receive full credit.