Math 299 **Midterm 2 Review** **Nov 1, 2013**

Exam topics

1. Methods of proof
   (a) Direct proof
   (b) Proof of the contrapositive
   (c) Proof by contradiction
   (d) Proof by cases
   (e) Proof by induction
      - Might involve proof by working backward
2. Divisibility of integers
   (a) Definition of divisibility
   (b) Properties of divisibility
   (c) Division Lemma
   (d) Definition of prime and composite numbers
   (e) Definition of *greatest common divisor*
   (f) Definition of coprime numbers
   (g) Definition of **mod**

You should know and be able to apply the following theorems.

1. Fundamental Theorem of Arithmetic
2. Every integer greater than or equal to 2 is divisible by at least one prime.
3. If n is composite integer, then it has a factor less than or equal to $\sqrt{n}$.
4. The Euclidean Algorithm
5. Let $g = \gcd(a, b)$. Then $\exists x, y \in \mathbb{Z}$ such that $ax + by = g$.
6. **Euclid's Lemma** Suppose $n, a, b \in \mathbb{N} \setminus \{0\}$. If $n \mid ab$ and $\gcd(a, n) = 1$, then $n \mid b$.

Practice Problems

1. Prove that for any two sets A and B, $(A \cup B)^c = A^c \cap B^c$.
2. Prove the if $n|a$ then $n|a+b \Leftrightarrow n|b$
3. Use Euclid's lemma to prove that if $gcd(m,n) = 1$ and $m|a$ and $n|a$ then the product $m \cdot n$ divides a.
4. Prove that if a, b are relatively prime, then $\forall c \in \mathbb{Z}, \exists x, y \in \mathbb{Z}$ such that $ax+by = c$.
5. Prove that $gcd(a+3b, b) \leq gcd(a, b+7a)$ for all $a, b \in \mathbb{Z}$ by using the definitions of divisibility and GCD only.
6. Use *proof by induction* to show that $5^{2k} - 1$ is divisible by 4 for all $k \in \mathbb{N}$.
7. Let $n \in \mathbb{N}$.

   (a) Use induction to show that exactly one element of the set $\{n, n+1, n+2, n+3\}$ is divisible by 4.

   (b) Use the Division Lemma to show that exactly one element of the set $\{n, n+1, n+2, n+3\}$ is divisible by 4.

8. Let $x \in \mathbb{Z}$.

   (a) Prove that $x^2 + x$ is even.

   (b) Prove that $(x^2 + x)/2$ is divisible by x if and only if x is odd.

   (c) Prove that $(x^2 + x)/2$ is divisible by $x + 1$ if and only if x is even.

9. (Houston 26.7 (iii)) Show that if $x^2 - 3x + 2 < 0$, then $1 < x < 2$.
10. (Houston 27.23 (v)) Prove that every common divisor of $a, b \in \mathbb{Z}$ is a divisor of $\gcd(a, b)$.
11. Let $a, b, c \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$, then $\gcd(a, bc) = 1$.
12. Recall that the Fibonacci numbers are defined by $F_1 = 1, F_2 = 1$, and

$$F_{n+1} = F_{n-1} + F_n, \qquad n \geq 2.$$

   (a) Prove that for all $n \in \mathbb{N}$, $\sum_{i=1}^{n} F_i = F_{n+2} - 1$.

   (b) Prove that every natural number can be written as the sum of distinct Fibonacci numbers. (This is a harder problem. Hint: use strong induction).

13. Let $a, b, c, d \in \mathbb{Z}$ with a and b nonzero. Prove that if $ab \nmid cd$, then $a \nmid c$ or $b \nmid d$.
14. Let x be an irrational real number. Prove that either x^2 or x^3 is irrational.

Note: These are just some sample problems. You have multiple examples to practice on from the lecture notes. Look at and review your lecture notes and you can bring your own questions to discuss in class.