Math 299

Methods of Proof (Solutions)

For each of the following statements, determine which of the following methods of proof is most appropriate, and then prove the statement.

Proof Methods: *direct proof, contrapositive, contradiction, proof by cases.*

STATEMENT 1. Let $n \in \mathbb{Z}$. If $n^2 + 6n + 5$ is even, then n is odd.

Proof: We prove this by contrapositive. Suppose $n = 2k$ for some $k \in \mathbb{Z}$. Then

$$n^2 + 6n + 5 = 4k^2 + 12k + 5 = 2(2k^2 + 6k + 2) + 1$$

is odd. Q.E.D.

STATEMENT 2. The sum of two rational numbers is rational.

Proof: We prove this directly. Let $p/q, m/n \in \mathbb{Q}$ be rational numbers. Note that this implies $q, n \neq 0$. Then

$$\frac{p}{q} + \frac{m}{n} = \frac{pn + qm}{qn}.$$

Since $pn + qm, qn \in \mathbb{Z}$, and since $qn \neq 0$, it follows that the right-hand side is rational. Q.E.D.

STATEMENT 3. The difference of two rational numbers is rational.

Proof: We prove this directly. Write

$$\frac{p}{q} - \frac{m}{n} = \frac{p}{q} + \frac{-m}{n}.$$

Then this follows from Statement 2. Q.E.D.

STATEMENT 4. Let $n \in \mathbb{Z}$. Then $n^2 + 2$ is not divisible by 4.

Solutions to come later.

STATEMENT 5. Every integer greater than 1 is divisible by at least one prime. (An integer is **prime** if it is greater than 1 and if it is only divisible by 1 and itself.)

This is extra credit.

STATEMENT 6. If the product of two real numbers is greater than 100, then at least one of the numbers is greater than 10.

(The statement should assume the real numbers are non-negative.)

Proof: We prove this by contrapositive. Suppose $0 \leq x, y \leq 10$. Multiply both sides of $x \leq 10$ to get

$$xy \leq 10y$$

(this uses $y \geq 0$). Similarly, multiply both sides of $y \leq 10$ to get

$$10y \leq 100.$$

Putting these together gives

$$xy \leq 100,$$

as desired. Q.E.D.

STATEMENT 7. If $x \in (0, 2)$, then $4x - 2 \in (-2, 6)$.

Proof: We prove this directly. Suppose $x \in (0, 2)$. Then

$$0 < x < 2$$

and so

$$0 < 4x < 8.$$

Subtract 2 to get

$$-2 < 4x - 2 < 6.$$

This is equivalent to $4x - 2 \in (-2, 6)$. Q.E.D.