Math 299 **Homework on Limits (Solutions)** **Fall 2013**

All references are to Beck's book.

1. Prove Proposition 10.14.

Proof: We want to prove that if (x_k) converges to L and to L', then $L = L'$.

For sake of contradiction, suppose this is not true, so $L \neq L'$. Let $\epsilon = \frac{1}{2}|L = L'|$. Then since (x_k) converges to L, there is some $N \in \mathbb{N}$ for which

$$|x_k - L| < \epsilon \tag{1}$$

for all $k \geq N$. Similarly, there is some $N' \in \mathbb{N}$ such that

$$|x_k - L'| < \epsilon \tag{2}$$

whenever $k \geq N'$. Let k be an integer greater than $\max\{N, N'\}$. Then by the triangle inequality we have

$$|L - L'| \leq |L - x_k| + |x_k - L'|.$$

Combining this with (1) and (2) gives

$$|L - L'| < 2\epsilon = |L - L'|,$$

which is a contradiction since there is no real number that satisfies $a < a$. □

2. Prove Proposition 10.11.

Proof: We want to show that $x = y$ if and only if, for every $\epsilon > 0$, we have $|x - y| < \epsilon$.

One direction is easy: If $x = y$ and $\epsilon > 0$ then $|x - y| = 0 < \epsilon$, as desired.

We prove the converse by proving the contrapositive: Suppose $x \neq y$, and set $\epsilon = \frac{1}{2}|x - y|$. Then $\epsilon > 0$, but it is not true that $|x - y| < \epsilon$. □

3. Prove Proposition 10.16.

Proof: Assume $\lim_{k\to\infty} x_k = L$. We want to prove that $\lim_{k\to\infty} x_{k+1} = L$. Let $\epsilon > 0$.

By assumption, there is some $N' \in \mathbb{N}$ such that

$$|x_{k'} - L| < \epsilon \qquad \forall k' \geq N'. \tag{3}$$

Let $N := N' - 1$. Then if $k \geq N$ we have $k + 1 \geq N'$ and so it follows from (3) applied to $k' = k + 1$ that

$$|x_{k+1} - L| < \epsilon,$$

as desired. □

4. Prove that if $x > 1$, then $\lim_{k\to\infty} x^k$ diverges. (See Prop. 10.17 and 10.18.)

Proof: Let $x > 1$. We want to show that for every $B \in \mathbb{R}$ there is some $N \in \mathbb{N}$ such that if $k \geq N$ is an integer, then $x^k \geq B$.

To prove this, let $B \in \mathbb{R}$. If $B \leq 0$ then we automatically have $x^k \geq B$, since $x^k \geq 0$. So we may assume $B > 0$. Then take N to be an integer greater than $\ln(B)/\ln(x)$ (we can take the natural log of B and x since they are both positive; also note that $\ln(x) \neq 0$ since $x \neq 1$). Then if $k \geq N$ we have

$$\ln(B)/\ln(x) \leq k.$$

Since $x > 1$ it follows that $\ln(x) > 0$ and so

$$\ln(B) \leq k \ln(x) = \ln(x^k).$$

The function $y \mapsto e^y$ is increasing, so this implies

$$B = e^{\ln(B)} \leq e^{\ln(x^k)} = x^k,$$

as desired. □

5. Prove Proposition 10.22.

Proof: Suppose $(x_k)_k$ is a convergent sequence. We want to show that $(x_k)_k$ is bounded. That is, we want to show that there is some $B > 0$ such that $|x_k| \leq B$ for all k.

Since $(x_k)_k$ converges, there are $L \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $|x_k - L| < 1$ for all $k \geq N$. By the triangle inequality, this implies

$$|x_k| \leq 1 + |L|$$

whenever $k \geq N$. Note that the set $\{|x_k| : 1 \leq k \leq N-1\}$ is finite, so its maximum exists. Call this maximum M. Then we take B to be $1 + |L|$ or M, whichever is largest. Then $|x_k| \leq B$ for all $k \in \mathbb{N}$. □

6. Prove $\lim_{k\to\infty} \frac{3k+1}{k} = 3$ using the $\epsilon - \delta$ definition of the limit.

Proof: Let $\epsilon > 0$. Then take $N > 1/\epsilon$. If $k \geq N$ then

$$\left|\frac{3k+1}{k} - 3\right| = \left|\frac{1}{k}\right| = \frac{1}{k} \leq \frac{1}{N} < \epsilon,$$

as desired. □

7. Define a sequence $(x_n)_{n=1}^{\infty}$ recursively by $x_1 = 1$, and $x_{n+1} = \frac{1}{2}x_n + 1$ for $n \geq 1$. Use Theorem 10.19 to show that $(x_n)_n$ converges. *Hint: Show this sequence is bounded and increasing.*

Proof: By the monotone convergence theorem, it suffices to show that this sequence is bounded above and increasing.

We will use induction to prove it is bounded above by 2. The base case is obviously true, so assume $x_n \le 2$. Then $x_{n+1} = \frac{1}{2}x_n + 1 \le \frac{1}{2}2 + 1 = 2$, as desired.

Now we show that it is increasing. We just saw $x_n \le 2$. This implies $\frac{1}{2}x_n \le 1$. Writing $\frac{1}{2}x_n = x_n - \frac{1}{2}x_n$, this gives $x_n - \frac{1}{2}x_n \le 1$, and so

$$x_n \le 1 + \frac{1}{2}x_n = x_{n+1}.$$

Since this is true for all n, it follows that $(x_n)_n$ is increasing. $\square$

8. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

Suppose $(x_k)_k$ is a sequence that converges to 0. Prove $\lim_{k\to\infty} f(x_k) = 0$.

Proof: Let $\epsilon > 0$. Since $(x_k)_k$ converges to 0, there is some N for which $|x_k| < \epsilon$ whenever $k \ge N$. It follows that if $k \ge N$, we have

$$|f(x_k) - 0| \le |x_k| < \epsilon.$$

Hence, $\lim_{k\to\infty} f(x_k) = 0$. $\square$