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1. Finish the proof from class of the monotone convergence theorem by showing that if $(x_n)_n$ is a sequence that is bounded below and decreasing, then $(x_n)_n$ converges.

Proof: Assume $(x_n)_n$ is bounded below and decreasing. Since it is bounded below, the infimum $L := \inf\{x_n\}$ exists. We will show $\lim_n x_n = L$. Let $\epsilon > 0$. Then $L + \epsilon > L$ and so $L + \epsilon$ cannot be a lower bound for $\{x_n\}$. It follows that there is some N such that $x_N < L + \epsilon$. Since $(x_n)_n$ is decreasing, if $n \geq N$ then $x_n \leq x_N$, and so $x_n < L + \epsilon$. Rearranging this, and using the fact that $L \leq x_n$ for all n, gives

$$|x_n - L| = x_n - L < \epsilon$$

whenever $n \geq N$. Q.E.D.

2. Show that if $\lim_{n\to\infty} x_n = \infty$, then $(x_n)_n$ diverges. Recall that, by definition, $\lim_{n\to\infty} x_n = \infty$ if for all $B \in \mathbb{R}$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \geq B$.

Assume $\lim_{n\to\infty} x_n = \infty$, and let $L \in \mathbb{R}$. It suffices to show that $(x_n)_n$ does *not* converge to L. Take $\epsilon = 1$, and let N be any natural number. Using the definition of $\lim_{n\to\infty} x_n = \infty$, with $B = L$, there is some N' such that $x_n \geq L + 1$ whenever $n \geq N'$. Then let n be any integer greater than N and N'. It follows that

$$|x_n - L| = x_n - L \geq 1 = \epsilon.$$

This proves that $(x_n)_n$ does not converge to L, as desired. Q.E.D.

3. Suppose that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge. Show that $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and its value equals $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

Proof: Assume that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, and let A, B be the values that they converge to. Let $s_k := \sum_{n=1}^{k} a_n$ and $t_k := \sum_{n=1}^{k} b_n$ denote the partial sums. The assumption that the series converge to A and B, respectively, is equivalent to saying that the sequences $(s_k)_k$ and $(t_k)_k$ converge to A and B, respectively. Then by Beck Proposition 10.23 (iii), we have

$$\lim_k s_k + t_k = \lim_k s_k + \lim_k t_k = A + B.$$

This is exactly the statement that $\sum_n a_n + b_n$ converges to $A + B$, since $s_k + t_k$ is the kth partial sum of $\sum_n a_n + b_n$. Q.E.D.