Math 299 **Homework 12 Solutions** **Fall 2013**

Problem 0. We will prove by induction that

$$\sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive $n \in \mathbb{N}$. The base case is

$$1^2 = \frac{1(2)(3)}{6}$$

which is obviously true. For the inductive step, assume

$$\sum_{x=1}^{k} x^2 = \frac{k(k+1)(2k+1)}{6}$$

for some positive $k \in \mathbb{N}$. Then

$$\sum_{x=1}^{k+1} x^2 = (k+1)^2 + \sum_{x=1}^{k} x^2 = (k+1)^2 + \frac{k(k+1)(2k+1)}{6},$$

where we have used the inductive hypothesis in the second equality. There is a factor of $k+1$ in each term on the right=hand side. Factoring this out gives

$$\begin{aligned}
\textstyle\sum_{x=1}^{k+1} x^2 &= (k+1)\left(k+1+\tfrac{k(2k+1)}{6}\right) \\
&= (k+1)\tfrac{6k+6+k(2k+1)}{6} \\
&= (k+1)2k^2+7k+66 \\
&= (k+1)(k+2)(2k+3)6 \\
&= (k+1)((k+1)+1)(2(k+1)+1)6,
\end{aligned}$$

as desired. Q.E.D.

Problem 1. We will use induction to prove that

$$1 \times 1! + 2 \times 2! + \ldots + n \times n! = (n+1)! - 1, \tag{1}$$

for all positive $n \in \mathbb{N}$. The base case is $1 \times 1! = 2! - 1$, which is obviously true. For the inductive step, assume (**??**) holds when $n = k$. Then

$$\begin{aligned}
1 \times 1! + 2 \times 2! + \ldots + k \times k! + (k+1) \times (k+1)! &= (k+1)! - 1 + (k+1) \times (k+1)! \\
&= (k+1)!(1+k+1) - 1 \\
&= (k+2)! - 1,
\end{aligned}$$

as desired. Q.E.D.

Problem 2. I will prove a slightly different problem than the on stated. Let $x_1 = 2$, and define $x_{n+1} := \frac{1}{3}x_n + 1$. We will prove that $x_n \leq 2$ for all $n \in \mathbb{N}$. The base case $x_1 = 2 \leq 2$ is obviously true. For the inductive step, assume $x_k \leq 2$ for some k. Then $x_{k+1} = \imath 3x_k + 1 \leq \frac{2}{3} + 1 \leq 2$. Q.E.D.

Problem 3. This is left for you.

Problem 4. We will prove that $n! \geq n^2$ for all $n \geq 4$. The base case is $4! \geq 4^2$, which is just $24 \geq 16$, so is true. Assume $k \geq 4$ is such that $k! \geq k^2$. Then

$$(k+1)! = (k+1)k! \geq (k+1)k^2.$$

We would be done if we could show that $k^2 \geq k+1$. This can be seen as follows:

$$k^2 \geq 4k \geq 2k = k + k \geq k + 1,$$

where we have used the fact that $k \geq 4$. Q.E.D.

Problem 6. This is left for you.

Problem 7. I did a version of this in class.