1. Decide whether the following statements are true or false. Prove the true ones and provide counterexamples for the false ones.

   (a) If a_n converges, then a_n/n converges.

   (b) If a_n converges and b_n is bounded, then $a_n b_n$ converges.

   (c) If $a_n \to \infty$ and $b_n \to -\infty$ as $n \to \infty$, then $a_n + b_n \to 0$ as $n \to \infty$.

   (d) If $a_n \to 0$ and $b_n \to 1$ as $n \to \infty$, then $b_n/a_n \to \infty$ as $n \to \infty$.

2. Suppose that $\{a_n\}$ is bounded. Prove that $a_n/n^k \to 0$, as $n \to \infty$ for all $k \in \mathbb{N}$.

3. Using the formal definition of the limit proof that if $\lim_{n\to\infty} a_n = 1$ then $\lim_{n\to\infty} \frac{a_n^2 - e}{a_n} = 1 - e$.

4. (AC) Let S be the set of all functions $f : \mathbb{N} \to \mathbb{N}$. Define a relation on S by letting $f \sim g$ if and only if $f(n) = g(n)$ for infinitely many n. Is this an equivalence relation? If so describe the equivalence classes.

5. (AC) Prove (assuming basic results of calculus) that $\int_0^\infty x^n e^{-x} dx = n!$.

6. (AC) For a function $f : \mathbb{R} \to \mathbb{R}$, define $\lim_{x\to c} f(x) = L$ to mean that $\forall \epsilon > 0\ \exists\, \delta > 0$ such that $\forall\, x \in \mathbb{R}, |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$s(x) = \begin{cases} 0 & : x \leq 0 \\ 1 & : x > 0 \end{cases}$$

   Prove by negating the definition of limit that it is not true that $\lim_{x\to 0} s(x) = 0$.

7. (a) Use a multiplication table to find all values $a \in \mathbb{Z}_7$ for which the equation

$$x^2 = a$$

   has a solution $x \in \mathbb{Z}_7$. For each such a, list all of the solutions x.

   (b) Find all solutions $x \in \mathbf{Z}_7$ to the equation $x^2 + \overline{2}x + \overline{6} = \overline{0}$.

8. Use quantifiers to express what it means for a sequence $(x_n)_{n\in\mathbb{N}}$ to *diverge*. You cannot use the terms *not* or *converge*.

9. Suppose $A, B \subseteq \mathbb{R}$ are bounded and non-empty. Show that $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

10. Suppose $S \subseteq \mathbb{R}$ is bounded and non-empty. Define a new set $3S$ by $3S = \{3x \mid x \in S\}$. Show that $\sup(3S) = 3\sup(S)$.

11. Let A, B be sets, and suppose there is a surjection $f : A \to B$. Prove that there is an injection $g : B \to A$.

12. (a) Define $x \in \mathbb{R}$ to be a **linear algebraic number** if there are integers $a, b \in \mathbb{Z}$, with $a \neq 0$, such that $ax + b = 0$. Prove that the set of linear algebraic numbers is countable. *Hint: Construct an injection into the set* $\mathbb{Z}^2$.

    (b) Define $x \in \mathbb{R}$ to be a **quadratic algebraic number** if there are integers $a, b, c \in \mathbb{Z}$, with $a \neq 0$, such that $ax^2 + bx + c = 0$. Prove that the set of quadratic algebraic numbers is countable. *Hint: Construct an injection into the set* $\mathbb{Z}^3$.

13. Use the formal definition of limit to prove the following.

    (a) $\lim_{n\to\infty} \frac{n^2+3}{2n^3-4} = 0$

    (b) $\lim_{n\to\infty} \frac{4n-5}{2n+7} = 2$

    (c) $\lim_{n\to\infty} \frac{n^3-3n}{n+5} = +\infty$

    (d) $\lim_{n\to\infty} \frac{n^2-7}{1-n} = -\infty$

14. For each of the following, determine if $\sim$ defines an equivalence relation on the set S. If it does, prove it and describe the equivalence classes. If it does not, explain why.

    (a) $S = \mathbb{R} \times \mathbb{R}$. For (a, b) and $(c, d) \in S$, define $(a, b) \sim (c, d)$ if $3a + 5b = 3c + 5d$.

    (b) $S = \mathbb{R}$. For $a, b \in S$, $a \sim b$ if $a < b$.

    (c) $S = \mathbb{Z}$. For $a, b \in S$, $a \sim b$ if $a \mid b$.

    (d) $S = \mathbb{R} \times \mathbb{R}$. For (a, b) and $(c, d) \in S$, define $(a, b) \sim (c, d)$ if $\lceil a \rceil = \lceil c \rceil$ and $\lceil b \rceil = \lceil d \rceil$. Here $\lceil x \rceil$ is the smallest integer greater than or equal to x.

15. Consider Z_n.

    (a) Under what conditions on n does every nonzero element have a multiplicative inverse? How about an additive inverse?

    (b) Does every nonzero element have a multiplicative inverse in Z_{21}?

    (c) Does 5 have a multiplicative inverse in Z_{21}? Explain why or why not. If it does, find 5^{-1}.

    (d) Solve the equation $5x - 14 = 19$ in $\mathbb{Z}_{21}$.

16. Let $A = \{a, b, c\}$ and $B = \{a, x\}$. List all elements of

    (a) $A \cup B$

    (b) $A \cap B$

    (c) $A \backslash B$

(d) $A \times B$

(e) Power set of A

17. Let $S(n) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid max\{x, y\} = n\}$. Prove that $S(3) \cap S(5)$ is the empty set.

18. Let A and B be sets with n elements. Show that any injective function from A to B is surjective as well using induction on n.

19. Let $f : \mathbb{N} \to \mathbb{N}$, given by $f(n) = |n - 4|$.

    (a) Prove that f is surjective

    (b) Prove that f is not injective

20. Let $f : A \to B$ and $g : B \to A$ be functions satisfying $f(g(x)) = x$ for all $x \in B$. Prove that f is surjective.

21. Let X be a set with n elements and $B = \{p, q\}$. Find the number of surjective functions from X to B.

22. Describe a concrete bijection from $\mathbb{N}$ to $\mathbb{N} \times \{1, 2, 3\}$. Briefly tell why it is injective and surjective.

23. Make a truth table for not $(A \vee B) \implies A \wedge B$. Find a shorter logically equivalent expression.

24. Find the negations of the following statements:

    (a) $(A \vee B) \wedge (B \vee C)$

    (b) $A \implies (B \wedge C)$

    (c) $\forall x \;\; \exists y \;\; (P(x) \vee (\text{not } Q(y)))$