1. Decide whether the following statements are true or false. Prove the true ones and provide counterexamples for the false ones.

   (a) If a_n converges, then a_n/n converges.

Solution. This seems reasonable so we will try to prove that it is true. In fact it would make sense that if a_n converges then a_n/n should converge to 0. So this is what we will try to show.

Goal: to find good N_ϵ so that:
$$n \geq N_\epsilon \implies \left|\frac{a_n}{n} - 0\right| < \epsilon$$

So now please consider:

$$\begin{aligned} \left|\frac{a_n}{n} - 0\right| &= \left|\frac{a_n - a}{n} + \frac{a}{n}\right| && \text{(Clever splitting of 0)} \\ &\leq \left|\frac{a_n - a}{n}\right| + \left|\frac{a}{n}\right| && \text{(Triangle Ineq.)} \\ &= |a_n - a|\frac{1}{n} + |a|\frac{1}{n} && (n \text{ is } +) \end{aligned}$$

Here is where we need to make our good choice for N_ϵ Let's choose $N_\epsilon = \max\{2, N'_\epsilon, N''_\epsilon\}$ where:

By the definition of a_n converging we can choose N'_ϵ so that :

$$n \geq N'_\epsilon \implies \boxed{|a_n - a| < \epsilon}$$

By since $\dfrac{1}{n} \to 0$ we can choose N''_ϵ so that :

$$n \geq N''_\epsilon \implies \left|\frac{1}{n} - 0\right| = \boxed{\frac{1}{n} < \frac{\epsilon}{2|a|}}$$

And therefore we get if $n \geq N_\epsilon$ then:

$$\begin{aligned} \left|\frac{a_n}{n} - 0\right| &\leq |a_n - a|\frac{1}{n} + |a|\frac{1}{n} && \text{(From before)} \\ &< (\epsilon)\frac{1}{(2)} + |a|\left(\frac{\epsilon}{2|a|}\right) && \text{(Substitutions)} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} && \text{(Simplify)} \end{aligned}$$

And so,
$$\left|\frac{a_n}{n} - 0\right| < \epsilon$$

So therefore by the definition of convergence $\dfrac{a_n}{n}$ **converges** (to 0). And so this statement is $\boxed{\text{TRUE}}$.

Lastly I should note that the choosing of N_ϵ takes some time and lots of scrap paper to work out. There is an easier way! If you use the theorem that says $\boxed{\text{Every convergent sequence is bounded}}$ it is very straightforward. Try it out (Or look at the proof of 2)!

(b) If a_n converges and b_n is bounded, then $a_n b_n$ converges.

Solution. Let's review with the definition of bounded:

> **Definition:** $\{x_n\}$ is **bounded** $\Leftrightarrow$ $\exists\, M > 0$ such that $x_n \leq M \;\forall\, n \in \mathbb{N}$

Let's try for a counterexample. Let's take some vary simple choices:

$a_n = 1 = \{1, 1, 1, \ldots\}$

$b_n = (-1)^n = \{-1, 1, -1, \ldots\}$

Then a_n obviously converges (to 1) and b_n is bounded by $M = 1$. and yet:

$a_n b_n = 1(-1)^n = (-1)^n$ which does not converge.

And so this statement is $\boxed{\text{FALSE}}$.

(c) If $a_n \to \infty$ and $b_n \to -\infty$ as $n \to \infty$, then $a_n + b_n \to 0$ as $n \to \infty$.

Solution. This should not be true. because things can head to $\pm\infty$ at different rates. Let's try for a counter-example. Let's choose:

$a_n = 2n$

$b_n = -n$

Then we can see that these meet are criteria of $a_n \to \infty$ and $b_n \to -\infty$ and yet:

$$a_n + b_n = 2n + (-n) = \boxed{n \to \infty \neq 0}$$

And so this statement is $\boxed{\text{FALSE}}$.

(d) If $a_n \to 0$ and $b_n \to 1$ as $n \to \infty$, then $b_n/a_n \to \infty$ as $n \to \infty$.

Solution. The claim is that this is FALSE and this will be because we could have $a_n \to 0$ and yet $a_n < 0\ \forall n$ which would yield $b_n/a_n \to -\infty$.

So let's take

$b_n = 1 \to 1$

$a_n = -\dfrac{1}{n} \to 0$

Then we can look at:

$b_n/a_n = 1/\left(-\dfrac{1}{n}\right) = \boxed{-n \to -\infty \neq \infty}$

And so this statement is $\boxed{\text{FALSE}}$.

Let's quickly take some practice formally showing that $-n \to -\infty$. Let's first recall a definition of what we mean by a sequence $x_n \to \pm\infty$.

Definition: $x_n \to \infty$ means:
$$\forall M \in \mathbb{R},\ \exists N_M \in \mathbb{N} \text{ such that: } n \geq N_M \implies x_n > M.$$

Definition: $x_n \to -\infty$ means:
$$\forall M \in \mathbb{R},\ \exists N_M \in \mathbb{N} \text{ such that: } n \geq N_M \implies x_n < M.$$

So it is now clear what we have to do:

Goal: to choose a good N_M so that:
$$n \geq N_M \implies -n < M$$

Since $M \in \mathbb{R}$ we need to do a little extra work to make sure that we choose $N_M \in \mathbb{N}$ but still it is not so bad.

Let's try: $N_M = \lceil |M| \rceil + 1$ (the ceiling function where we just round up of the absolute value.) So now

$n \geq \lceil |M| \rceil + 1 \implies -n \leq \lceil |M| \rceil - 1 < M$

And so we have $\boxed{n \geq N_M \implies -n < M}$ so by the definition: $\boxed{-n \to -\infty}$

Honestly though I had to sit and play with this for quite some time to see that $N_M = \lceil |M| \rceil + 1$ was a good choice. To help me I considered what if $M = -4.5$ or what if $M = 4.5$ to help me:

If $M = -4.5$ then we get
$N_M = \lceil |-4.5| \rceil + 1 = \lceil 4.5 \rceil + 1 = 5 + 1 = 6$ so we get $n \geq 6 \implies -n < -4.5$

Or if $M = 4.5$ then we get
$N_M = \lceil |4.5| \rceil + 1 = \lceil 4.5 \rceil + 1 = 5 + 1 = 6$ so we get $n \geq 6 \implies -n < 4.5$

Choosing nice specific examples like this can help craft good choices of N_M (or δ or N_ϵ) on occasion.

2. Suppose that $\{a_n\}$ is bounded. Prove that $a_n/n^k \to 0$, as $n \to \infty$ for all $k \in \mathbb{N}$.

Solution. The first step is to preform some simplifications. Let's apply the following theorem

Theorem:

$$x_n \to 0 \quad \Leftrightarrow \quad |x_n| \to 0$$

Which is a result of the Squeeze Theorem. So therefore $a_n/n^k \to 0 \Leftrightarrow |a_n|/n^k \to 0$.

Now let's apply the Squeeze Theorem directly which states:

Theorem: (Squeeze)

If $x_n \to a$ and $y_n \to a$ and if $x_n \leq w_n \leq y_n \quad \forall\, n \geq N_0$ then:

$$w_n \to a$$

Our desired thing to show converges is $|a_n|/n^k \to 0$. So let's make good choice for $x_n \to 0$ and $y_n \to 0$ so that the Squeeze Theorem will help us.

Since we have $0 \leq |a_n|/n^k$ we can take $\boxed{x_n = 0 \to 0}$. Now for y_n let's use the fact that a_n bounded, which means that:

$$\exists\, M > 0 \text{ such that } a_n \leq M \;\forall\, n \in \mathbb{N}$$

Also if $n, k \geq 1$ then $n \leq n^k \implies \dfrac{1}{n^k} \leq \dfrac{1}{n}$. Comboing these we have that $|a_n|/n^k \leq M/n^k \leq M/n$. So therefore we should choose $\boxed{y_n = M/n}$ which converges to 0 by the multiplicative rule of limits

$$\frac{M}{n} = M\frac{1}{n} \to M(0) = 0$$

By the Squeeze theorem we win!

3. Using the formal definition of the limit proof that if $\lim_{n\to\infty} a_n = 1$ then $\lim_{n\to\infty} \frac{a_n^2 - e}{a_n} = 1 - e$.

Solution. This would be much easier if we could use properties of limits. Let's try it out using the definition though:

Goal: to find good N_ϵ so that:

$$n \geq N_\epsilon \implies \left|\frac{a_n^2 - e}{a_n} - (1-e)\right| < \epsilon$$

We have in our tool belt that $a_n \to 1$ which means:

$$\forall \epsilon' > 0 \; \exists N'_\epsilon \in \mathbb{N} \text{ so that: } n \geq N'_\epsilon \implies |a_n - 1| < \epsilon' \qquad (**)$$

So let's rearrange $\left|\frac{a_n^2 - e}{a_n} - (1-e)\right|$ a bit and see if there is any natural progressions!

$$\begin{aligned}
\left|\frac{a_n^2 - e}{a_n} - (1-e)\right| &= \left|\frac{a_n^2 - e - a_n + ea_n}{a_n}\right| && \text{(Common Denominator)}\\
&= \left|\frac{a_n^2 + (e-1)a_n - e}{a_n}\right| && \text{(Rearrange to look like Quadratic)}\\
&= \left|\frac{(a_n + e)(a_n - 1)}{a_n}\right| && \text{(Factor)}\\
&= \left|\frac{a_n + e}{a_n}\right| |a_n - 1| && \text{(Property of } |\cdot|\text{)}
\end{aligned}$$

Here we need to employ some trick to bound $\left|\frac{a_n + e}{a_n}\right|$. Let's consider N''_ϵ so that
$n \geq N''_\epsilon \implies |a_n - 1| < .5 \implies .5 < a_n < 1.5$.
Now to make it look like $\left|\frac{a_n + e}{a_n}\right|$.

$$.5 < a_n < 1.5 \implies 0 < .5 + e < a_n + e < 1.5 + e \implies 0 < \frac{a_n + e}{a_n} < \frac{1.5 + e}{.5} = 3 + 2e \implies \boxed{\left|\frac{a_n + e}{a_n}\right| < 3 + 2e}$$

So let's consider choosing N'_ϵ from (**??**) so that:

$$n \geq N'_\epsilon \implies \boxed{|a_n - 1| < \frac{\epsilon}{3 + 2e}}$$

Finally taking $N_\epsilon = \max\{N'_\epsilon, N''_\epsilon\}$ gives us:

$$n \geq N_\epsilon \implies \left|\frac{a_n + e}{a_n}\right| |a_n - 1| < (3 + 2e)\left(\frac{\epsilon}{3 + 2e}\right) = \epsilon$$

Making us the winners!

4. (AC) Let S be the set of all functions $f : \mathbb{N} \to \mathbb{N}$. Define a relation on S by letting $f \sim g$ if and only if $f(n) = g(n)$ for infinitely many n. Is this an equivalence relation? If so describe the equivalence classes.

Solution. The relation is reflexive since for any $f : \mathbb{N} \to \mathbb{N}$, $f(n) = f(n)\ \forall\, n \in \mathbb{N}$ where $\mathbb{N}$ is an infinite set. The relation is symmetric since $=$ is symmetric. In fact, the relation is not transitive. To see why, consider the following functions. Let $f(n)$ be equal to 1 for n even and 2 for n odd. Let $g(n)$ be equal to 3 for n even and 2 for n odd. Let $h(n)$ be equal to 3 for n even and 4 for n odd. Then $f \sim g$ and $g \sim h$ but f and h agree nowhere.

5. (AC) Prove (assuming basic results of calculus) that $\int_0^\infty x^n e^{-x} dx = n!$.

Solution. We proceed by induction on $n \geq 0$. The base case holds since

$$\begin{aligned}
\int_0^\infty e^{-x} dx &= \lim_{b\to\infty} \int_0^b e^{-x} dx \\
&= \lim_{b\to\infty} -e^{-x}|_0^b \\
&= \lim_{b\to\infty} -e^{-b} + 1 \\
&= 1 \\
&= 0!
\end{aligned}$$

To show that the inductive step holds, we integrate by parts with $u = x^n, dv = e^{-x} dx$. We have

$$\begin{aligned}
\int_0^\infty x^n e^{-x} dx &= \lim_{b\to\infty} \int_0^b x^n e^{-x} dx \\
&= \lim_{b\to\infty} \left[[x^n(-e^{-x})]_0^b - \int_0^b n x^{n-1} [-e^{-x}] dx \right] \\
&= 0 + n \int_0^\infty x^{n-1} e^{-x} dx \\
&= n(n-1)! \qquad \text{by the inductive hypothesis} \\
&= n!.
\end{aligned}$$

Thus by induction, the identity holds for all $n \geq 0$.

6. (AC) For a function $f : \mathbb{R} \to \mathbb{R}$, define $\lim_{x\to c} f(x) = L$ to mean that $\forall \epsilon > 0\ \exists\, \delta > 0$ such that $\forall\, x \in \mathbb{R}, |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$s(x) = \begin{cases} 0 & : x \leq 0 \\ 1 & : x > 0 \end{cases}$$

Prove by negating the definition of limit that it is not true that $\lim_{x\to 0} s(x) = 0$.

Solution. The negation of $\lim\limits_{x\to 0} s(x) = 0$ is

$$\exists\, \epsilon > 0 \text{ such that } \forall\, \delta > 0, \exists x \in \mathbb{R} \text{ such that } |x - 0| < \delta \text{ and } |s(x) - 0| \geq \epsilon$$

Letting $\epsilon = 1/2$, then $\forall\, \delta > 0, \delta/2 \in \mathbb{R}$ with $|\delta/2 - 0| = \delta/2 < \delta$ and $|s(\delta/2) - 0| = |1 - 0| = 1 \geq 1/2$. Thus the limit is not equal to 0.

7. (a) Use a multiplication table to find all values $a \in \mathbb{Z}_7$ for which the equation

$$x^2 = a$$

has a solution $x \in \mathbb{Z}_7$. For each such a, list all of the solutions x.

Solution. We only need to look at the diagonal of the multiplication table for $\mathbb{Z}_7$. Then the equation $x^2 = a$ has a solution $x \in \mathbb{Z}_7$ if and only if $a \in \left\{\overline{0}, \overline{1}, \overline{2}, \overline{4}\right\}$. When $a = \overline{0}$, the only solution is $x = \overline{0}$. When $a = \overline{1}$, the solutions are $x = \overline{1}$ and $x = \overline{6}$. When $a = \overline{2}$, the solutions are $x = \overline{3}$ and $x = \overline{4}$. When $a = \overline{4}$, the solutions are $x = \overline{2}$ and $x = \overline{5}$.

(b) Find all solutions $x \in \mathbf{Z}_7$ to the equation $x^2 + \overline{2}x + \overline{6} = \overline{0}$.

Solution. Adding $\overline{2}$ to both sides, the given equation is equivalent to $x^2 + \overline{2}x + \overline{1} = \overline{2}$. We can factor the left-hand side to get

$$(x + \overline{1})^2 = \overline{2}.$$

It follows from part (a) that $x + \overline{1} = \overline{3}$ or $x + \overline{1} = \overline{4}$, and hence $x = \overline{2}$ or $x = \overline{3}$.

8. Use quantifiers to express what it means for a sequence $(x_n)_{n\in\mathbb{N}}$ to *diverge*. You cannot use the terms *not* or *converge*.

Solution. A sequence $(x_n)_{n\in\mathbb{N}}$ *diverges* if for every $L \in \mathbb{R}$ there is some $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there is some natural number $n \geq N$ for which $|x_n - L| \geq \epsilon$. In terms of quantifiers this is

$$\forall L \in \mathbb{R}\ \exists \epsilon > 0\ \forall N \in \mathbb{N}\ \exists n \geq N, |x_n - L| \geq \epsilon.$$

9. Suppose $A, B \subseteq \mathbb{R}$ are bounded and non-empty. Show that $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

Solution. First note that since A and B are both bounded and non-empty, the same is true of $A \cup B$ and so $\sup(A \cup B) \in \mathbb{R}$ exists. It follows immediately from Beck Proposition 8.50 that $\sup(A \cup B) \geq \max\{\sup(A), \sup(B)\}$, since A and B are both subsets of $A \cup B$. We need to prove the reverse inequality. For sake of contradiction, suppose $\sup(A \cup B) > \max\{\sup(A), \sup(B)\}$. Then neither $\sup(A)$, nor $\sup(B)$ can be an upper bound for $A \cup B$. So there is some $x \in A \cup B$ with $x > \sup(A)$ and $x > \sup(B)$. But $x \in A \cup B$ implies that $x \in A$ or $x \in B$, so this cannot happen.

10. Suppose $S \subseteq \mathbb{R}$ is bounded and non-empty. Define a new set $3S$ by $3S = \{3x \mid x \in S\}$. Show that $\sup(3S) = 3\sup(S)$.

Solution. First we will show $\sup(3S) \leq 3\sup(S)$. Let $y \in 3S$. Then $y = 3x$ for some $x \in S$. Since $\sup(S)$ is an upper bound for S, we have $x \leq \sup(S)$. This gives $y = 3x \leq 3\sup(S)$. Since this is true for all $y \in 3S$, it follows that $3\sup(S)$ is an upper bound for the set $3S$. Since $\sup(3S)$ is the *least* upper bound, we must have $\sup(3S) \leq 3\sup(S)$.

To prove the reverse inequality, let $x \in S$. Then $3x \in 3S$ and so $3x \leq \sup(3x)$. Equivalently, $x \leq 3^{-1}\sup(3x)$. It follows that $3^{-1}\sup(3x)$ is an upper bound for S and so $\sup(S) \leq 3^{-1}\sup(3x)$, since $\sup(S)$ is the *least* upper bound. Multiplying both sides by 3 proves the result.

11. Let A, B be finite sets, and suppose there is a surjection $f : A \to B$. Prove that there is an injection $g : B \to A$ such that $f \circ g : B \to B$ is the identity function.

Solution. Let $b \in B$, we want to define $g(b) \in A$. Since f is surjective, it follows that $f^{-1}(b)$ is a non-empty set. Let $a \in f^{-1}(b)$ be any element of this set, and declare $g(b) = a$. We obviously have that $f \circ g$ is the identity, since $f(g(b)) = f(a) = b$ for all $b \in B$. To show g is injective, suppose there are $b, b' \in B$ with $g(b) = g(g')$. Let $a = g(b)$ denote this common value. Then by construction of g, we have $a \in f^{-1}(b)$ and $a \in f^{-1}(b')$. Applying f to a therefore gives $f(a) = b$ and $f(a) = b'$, and so $b = b'$.

12. (a) Define $x \in \mathbb{R}$ to be a **linear algebraic number** if there are integers $a, b \in \mathbb{Z}$, with $a \neq 0$, such that $ax + b = 0$. Prove that the set of linear algebraic numbers is countable. *Hint: Construct an injection into the set* $\mathbb{Z}^2$.

Solution. Since $\mathbb{Z}^2$ is countable, it suffices to show that there is an injection from the set of linear algebraic numbers into $\mathbb{Z}^2$. Given a linear algebraic number x, there are $a, b \in \mathbb{Z}$ with $ax + b = 0$, and $a \neq 0$. (Note that there will be multiple choices of a, b for which $ax + b = 0$. However, we can choose a, b canonically be requiring $a > 0$ and $\gcd(a, b) = 1$.) Then we define the function by sending x to these values of a, b. To see that this function is injective, suppose there are $a, b \in \mathbb{Z}$ with $a \neq 0$ and $ax + b = ay + b$ for some $x, y \in \mathbb{R}$. Then obviously $x = y$, since $a \neq 0$. (Note that the linear algebraic numbers are exactly the rational numbers, and we are just proving that $\mathbb{Q}$ is countable.)

(b) Define $x \in \mathbb{R}$ to be a **quadratic algebraic number** if there are integers $a, b, c \in \mathbb{Z}$, with $a \neq 0$, such that $ax^2 + bx + c = 0$. Prove that the set of quadratic algebraic numbers is countable. *Hint: Construct an injection into the set* $\mathbb{Z}^3$.

Solution. We will show that there is an injective function from the set of quadratic algebraic numbers into $\mathbb{Z}^3$. Given a quadratic algebraic number x, there are $a, b, c \in \mathbb{Z}$ with $ax^2 + bx + c = 0$, and $a \neq 0$. As in part (a), the basic idea is to send x to $(a, b, c) \in \mathbb{Z}^3$. However, there is an extra feature of this problem that makes it a little more difficult, and this is coming from the fact that most quadratic equations have 2 solutions (this will mean that our function will not be injective, unless we are careful). Here is a way to define the function so that it is injective. Let x be as above. If there is some $x' \in \mathbb{R}$ with $a(x')^2 + bx' + c = 0$ and $x \neq x'$, then without loss of generality we may suppose $x' < x$. Send x to (a, b, c) where $a > 0$ and a, b, c have no common divisors, and send x' to $(-a, -b, -c)$. (Note that by the quadratic formula there can be at most two solutions of $ax^2 + bx + c = 0$.)
To see that this function is injective, suppose x_1 and x_2 are both sent to (a, b, c). This implies that x_1, x_2 are both solutions to the equation $ax^2 + bx + c = 0$. If $a > 0$, then by construction of our function, x_1 and x_2 are both equal to the solution with the maximum values and so $x_1 = x_2$. If $a < 0$, then x_1 and x_2 are both equal to the solution with the smallest value and so $x_1 = x_2$.

13. Use the formal definition of limit to prove the following.

(a) $\displaystyle\lim_{n\to\infty} \frac{n^2 + 3}{2n^3 - 4} = 0$

Solution. Let $\varepsilon > 0$ be given, aribitrary. Define $N = \max\{3, \frac{2}{\varepsilon} + 2\}$. Let $n \geq N$, $n \in \mathbb{N}$ be arbitrary. Then,

$$\begin{aligned}
\left|\frac{n^2+3}{2n^3-4} - 0\right| &= \frac{n^2+3}{2n^3-4} && \text{(since } n \geq 3\text{)} \\
&\leq \frac{n^2+3n^2}{2n^3-4n^2} && \text{(We increased the numerator and decreased the denominator,} \\
& && \text{keeping in mind } 2n^3 - 2n^2 > 0, \text{ as } n > 2\text{)} \\
&= \frac{2}{n-2} && \text{(Factor and cancel out common terms)} \\
&\leq \frac{2}{N-2} && (n \geq N) \\
&\leq \varepsilon && (N \geq \frac{2}{\varepsilon} + 2)
\end{aligned}$$

Thus, $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$ with $n \in \mathbb{N}$, $\left|\frac{n^2+3}{2n^3-4} - 0\right| < \varepsilon$. Thus, indeed $\lim_{n\to\infty} \frac{n^2+3}{2n^3-4} = 0$.

(b) $\lim_{n\to\infty} \frac{4n-5}{2n+7} = 2$

Solution. Let $\varepsilon > 0$ be given, aribitrary. Define $N = \frac{1}{\varepsilon}$. Let $n \geq N$, $n \in \mathbb{N}$ be arbitrary. Then,

$$\begin{aligned}
\left|\frac{4n-5}{2n+7} - 2\right| &= \frac{19}{2n+7} && \text{(since } n > 0\text{)} \\
&< \frac{19}{2n} && \text{(We decreased the denominator,} \\
&< \frac{1}{n} \\
&\leq \frac{1}{N} && (n \geq N) \\
&= \varepsilon && (N = \frac{1}{\varepsilon})
\end{aligned}$$

Thus, $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$ with $n \in \mathbb{N}$, $\left|\frac{4n-5}{2n+7} - 2\right| < \varepsilon$. Thus, indeed $\lim_{n\to\infty} \frac{4n-5}{2n+7} = 0$.

(c) $\lim_{n\to\infty} \frac{n^3-3n}{n+5} = +\infty$

Solution. Let $M > 0$ be given, aribitrary. Define $N = \sqrt{6M+3}$. Let $n \geq N$, $n \in \mathbb{N}$ be arbitrary. Then,

$$\begin{aligned}
\frac{n^3-3n}{n+5} &\geq \frac{n^3-3n}{n+5n} && \text{(since } n \geq 1\text{)} \\
&= \frac{n^3-3n}{6n} \\
&= \frac{n^2-3}{6} \\
&\geq \frac{N^2-3}{6} && (n \geq N) \\
&= M && (N = \sqrt{6M+3})
\end{aligned}$$

Thus, $\forall M > 0$, $\exists N$ such that $\forall n > N$ with $n \in \mathbb{N}$, $\frac{n^3-3n}{n+5} \geq M$. Thus, indeed $\lim_{n\to\infty} \frac{n^3-3n}{n+5} = +\infty$.

(d) $\lim_{n\to\infty} \frac{n^2-7}{1-n} = -\infty$

Solution. Let $M < 0$ be given, aribitrary. Define $N = 7 - M$. Let $n \geq N$, $n \in \mathbb{N}$ be arbitrary. Then,

$$\begin{aligned}
\frac{n^2-7}{1-n} &< \frac{n^2-7}{-n} && \text{(since } n > 7,\ \text{thus } n^2-7 > 0\text{)} \\
&\leq \frac{n^2-7n}{-n} && \text{(since the denominator is negative and } n > 7, \\
& && \text{decreasing the numerator, while still keeping it positive)} \\
&= 7-n \\
&\leq 7-N && (n \geq N) \\
&= M && (N = 7-M)
\end{aligned}$$

Thus, $\forall M < 0$, $\exists N$ such that $\forall n > N$ with $n \in \mathbb{N}$, $\frac{n^2-7}{1-n} \leq M$. Thus, indeed $\lim_{n\to\infty} \frac{n^2-7}{1-n} = -\infty$.

14. For each of the following, determine if $\sim$ defines an equivalence relation on the set S. If it does, prove it and describe the equivalence classes. If it does not, explain why.

(a) $S = \mathbb{R} \times \mathbb{R}$. For (a, b) and $(c, d) \in S$, define $(a, b) \sim (c, d)$ if $3a + 5b = 3c + 5d$.

Solution. The relation $\sim$ as defined above is indeed an equivalence relation, since it satisfies reflexivity, symmetry and transitivity, as shown below.

- Reflexivity: Let $(a,b) \in S$. Then $3a+5b = 3a+5b$, and therefore $(a,b) \sim (a,b)$.
- Symmetry: Let $(a,b),(c,d) \in S$ such that $(a,b) \sim (c,d)$. Then $3a+5b = 3c+5d$. This is equivalent to $3c+5d = 3a+5b$, which implies $(c,d) \sim (a,b)$.
- Transitivity: Let $(a,b),(c,d),(e,f) \in S$, such that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Then $3a+5b = 3c+5d$ and $3c+5d = 3e+5f$. By transitivity of equality for real numbers we have $3a+5b = 3e+5f$, and therefore $(a,b) \sim (e,f)$.

The equivalence classes are the lines $3x+5y=c$, i.e. each equivalence class is a line with slope $-\frac{3}{5}$ and the different equivalence classes have different $y-$intercepts (given by $\frac{c}{5}$).

(b) $S = \mathbb{R}$. For $a,b \in S$, $a \sim b$ if $a < b$.

Solution. The relation defined by $a \sim b$ if $a < b$ is not an equivalence relation, since it does not satisfy reflexivity. Namely, $a \nsim a$, since $a \nless a$.

(c) $S = \mathbb{Z}$. For $a,b \in S$, $a \sim b$ if $a \mid b$.

Solution. The relation defined by $a \sim b$ if $a \mid b$ is not an equivalence relation, since it does not satisfy symmetry. Namely, $a \sim b$ does not necessarily imply $b \sim a$. For example, $2 \mid 8$, but $8 \nmid 2$.

(d) $S = \mathbb{R} \times \mathbb{R}$. For (a,b) and $(c,d) \in S$, define $(a,b) \sim (c,d)$ if $\lceil a \rceil = \lceil c \rceil$ and $\lceil b \rceil = \lceil d \rceil$. Here $\lceil x \rceil$ is the smallest integer greater than or equal to x.

Solution. The relation $\sim$ as defined above is indeed an equivalence relation, since it satisfies reflexivity, symmetry and transitivity, as shown below.

- Reflexivity: Let $(a,b) \in S$. Then $\lceil a \rceil = \lceil a \rceil$ and $\lceil b \rceil = \lceil b \rceil$, and therefore $(a,b) \sim (a,b)$.
- Symmetry: Let $(a,b),(c,d) \in S$ such that $(a,b) \sim (c,d)$. Then $\lceil a \rceil = \lceil c \rceil$ and $\lceil b \rceil = \lceil d \rceil$. This is equivalent to $\lceil c \rceil = \lceil a \rceil$ and $\lceil d \rceil = \lceil b \rceil$, which implies $(c,d) \sim (a,b)$.
- Transitivity: Let $(a,b),(c,d),(e,f) \in S$, such that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Then $\lceil a \rceil = \lceil c \rceil$ and $\lceil b \rceil = \lceil d \rceil$, as well as $\lceil c \rceil = \lceil e \rceil$ and $\lceil d \rceil = \lceil f \rceil$. By transitivity of equality for real numbers we have $\lceil a \rceil = \lceil e \rceil$ and $\lceil b \rceil = \lceil f \rceil$, and therefore $(a,b) \sim (e,f)$.

The equivalence classes are squares in the plane $\mathbb{R}^2$ with sides parallel to the coordinate axes, in particular, they are sets of the form $(i, i+1] \times (j, j+1]$ (Cartesian product of intervals), where the ordered pair $(i,j) \in \mathbb{Z}^2$.

15. Consider Z_n.

    (a) Under what conditions on n does every nonzero element have a multiplicative inverse? How about an additive inverse?

    Solution. Every nonzero element in $\mathbb{Z}_n$ has a multiplicative inverse if n is prime. Indeed, if n is prime, $\gcd(n, m) = 1 \; \forall m \in \mathbb{Z}$ such that $0 < m < n$, and therefore by Bezout's Lemma there exist integers x, y such that $nx + my = 1$, thus $my \equiv 1 \mod n$, i.e. $\bar{m} \cdot \bar{y} = 1$, which implies that $\bar{m}^{-1} = \bar{y}$.

    Every element in $\mathbb{Z}_n$ does have an additive inverse $\forall n \in \mathbb{N}$.

    (b) Does every nonzero element have a multiplicative inverse in Z_{21}?

    Solution. No, one can check that $\bar{3}$ and $\bar{7}$ do not have multiplicative inverses in $\mathbb{Z}_{21}$.

    (c) Does 5 have a multiplicative inverse in Z_{21}? Explain why or why not. If it does, find 5^{-1}.

    Solution. One can express $\gcd(5, 21)$ in the form $5x + 21y$ for some integers x, y by applying the Euclidean Algorithm, $21 = 4 \cdot 5 + 1$, therefore $5 \cdot (-4) + 21 \cdot 1 = 1$, thus $\bar{5}^{-1} = \bar{17}$. (Note that the equivalence classes $\bar{-4} = \bar{17}$.)

    (d) Solve the equation $5x - 14 = 19$ in $\mathbb{Z}_{21}$.

    Solution. The equation $\bar{5}x - \bar{14} = \bar{19}$ is equivalent to $\bar{5}x = \bar{12}$, which, using that $\bar{5}^{-1} = \bar{17}$ yields $x = \bar{12} \cdot \bar{17} = \bar{15}$.

16. Let $A = \{a, b, c\}$ and $B = \{a, x\}$. List all elements of

    (a) $A \cup B$

    (b) $A \cap B$

    (c) $A \backslash B$

    (d) $A \times B$

    (e) Power set of A

    Solution. $A \cup B = \{a, b, c, x\}, \quad A \cap B = \{a\}, \quad A \backslash B = \{b, c\}$, $A \times B = \{(a, a), (a, x), (b, a), (b, x), (c, a), (c, x)\}$, Power set of A is $\{\emptyset, \ \{a\}, \ \{b\}, \ \{c\}, \ \{a, b\}, \ \{b, c\}, \ \{a, c\}, \ \{a, b, c\}\}$.

17. Let $S(n) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid max\{x, y\} = n\}$. Prove that $S(3) \cap S(5)$ is the empty set.

Solution. Assume the contrary: let $(a, b) \in S(3) \cap S(5)$. Then $max\{a, b\} = 3$ and $max\{a, b\} = 5$, but $3 \neq 5$, hence our assumption leads to a contradiction. Therefore the intersection is empty.

18. Let A and B be sets with n elements. Show that any injective function from A to B is surjective as well using induction on n.

Solution. *Base case,* $n = 1$. Since $A = \{a_1\}$ and $B = \{b_1\}$ have only one element each, there is only one function f, given by $f(a_1) = b_1$ which is injective ($f(x) = f(y)$ implies $x = y = a_1$ since A consists only of a_1) and surjective (b_1 is the image of a_1 under f).

Inductive Hypothesis. Assume A and B are sets with $n = k$ elements and that any injective function from A to B is surjective.

Inductive Step. Show that the same statement is true for $n = k + 1$: Let $A = \{a_1, ..., a_{k+1}\}$ and $B = \{b_1, ..., b_{k+1}\}$. Let $A' = A\backslash\{a_k + 1\}$ and $B' = B\backslash f(a_{k+1})$. Since f is injective, $f(A')$ does not contain $f(a_{k+1})$, hence f restricts to a function f' from A' to B', both of which have k elements. Claim: f' is injective (short proof), hence by the inductive hypothesis, f' is surjective. To check that f is surjective, if $b = f(a_{k+1})$, b is the image of a_{k+1}, otherwise, for any $b \in B$ with $b \neq f(a_{k+1})$, we have $b \in B'$, hence $b = f'(a_i)$ for some i, hence $b = f(a_i)$.

19. Let $f : \mathbb{N} \to \mathbb{N}$, given by $f(n) = |n - 4|$.

    (a) Prove that f is surjective

    (b) Prove that f is not injective

Solution. Given $y \in \mathbb{N}$, $f(y + 4) = |y + 4 - 4| = y$ since $y \geq 0$, which shows that f is surjective. $f(1) = 3 = f(7)$, hence f is not injective.

20. Let $f : A \to B$ and $g : B \to A$ be functions satisfying $f(g(x)) = x$ for all $x \in B$. Prove that f is surjective.

Solution. First attempt: Assume f is not surjective. Then there is a $b \in B$ such that there are no $a \in A$ with $f(a) = b$. Let $c = g(b)$, then $f(c) = f(g(b)) = b$ by assumption, hence we found a $c \in A$ with $f(c) = b$ which contradicts with the assumption.

Second attempt: Given $b \in B$, let $a = g(b)$, and compute $f(a) = f(g(b)) = b$ by assumption. Since b was arbitrary, this shows that f is surjective.

21. Let X be a set with n elements and $B = \{p, q\}$. Find the number of surjective functions from X to B.

Solution. There are $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ possible functions from X to B. Now list all functions that are not surjective (there are exactly two, tell which ones), hence we get $2^n - 2$.

22. Describe a concrete bijection from $\mathbb{N}$ to $\mathbb{N} \times \{1, 2, 3\}$. Briefly tell why it is injective and surjective.

Solution. By division lemma, given n, there is a unique q and r with $0 \leq r < 3$ with $n = 3 \cdot q + r$, we could define $f(n)$ by $f(n) = (q, r + 1)$. Then f has the inverse function given by $g(a, b) = 3 \cdot a + b$. Check that $f(g(a, b)) = (a, b)$ and $g(f(n)) = n$.

23. Make a truth table for not $(A \vee B) \implies A \wedge B$. Find a shorter logically equivalent expression.

Solution. Consider all possibilities for simultaneous truth values for A and B:

A	B	not $(A \vee B)$	$A \wedge B$	not $(A \vee B) \implies$ not A
T	T	F	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	F

We see that the only time the expression is False is when both A and B are False, hence this expression is logically equivalent to $A \vee B$.

24. Find the negations of the following statements:

(a) $(A \vee B) \wedge (B \vee C)$

(b) $A \implies (B \wedge C)$

(c) $\forall x \;\; \exists y \;\; (P(x) \vee (\text{not } Q(y)))$

Solution. $\text{not}((A \vee B) \wedge (B \vee C)) \equiv \text{not}(A \vee B) \vee \text{not}(B \vee C) \equiv (\text{not}A \wedge \text{not}B) \vee (\text{not}B \wedge \text{not}C)$

$\text{not}(A \implies (B \wedge C)) \equiv \text{not}(\text{not}A \vee (B \wedge C)) \equiv A \wedge \text{not}(B \wedge C) \equiv A \wedge (\text{not}B \vee \text{not}C)$

$\text{not}(\forall x \;\; \exists y \;\; (P(x) \vee (\text{not } Q(y)))) \equiv \exists x \;\; \forall y \;\; (\text{not}P(x)) \wedge Q(y)$