Math 153H **Recitation Worksheet #3** **Sep 16, 2014**

1. Solve the following initial value problems for $y = y(x)$.

(a) $\frac{dy}{dx} = e^{3x+y}, \qquad y(0) = 0.$

(b) $x^2 \frac{dy}{dx} = y^2, \qquad y(1) = 1.$

(c) $x^2 \frac{dy}{dx} = y^2, \qquad y(0) = 0.$ (Can you find more than one solution?[1])

---

[1]The reason most of the other equations only have one solution (as expected) is that they can be written as $dy/dx = F(x, y)$, where F is *continuous* at the initial condition. This is not the case for the initial value problem in part (c).

(d) $(1+x^2)\frac{dy}{dx}+xy=0, \qquad y(0)=2.$

(e) $\sec^3(x)\frac{dy}{dx}=\cos^2(y), \qquad y(0)=\pi/4.$

2. Consider the following differential equation

$$x'(t)+x(t)=e^t\cos(t). \tag{1}$$

In this problem we will show that there exists a unique solution x to this equation with the initial condition $x(0)=100$. We begin with *uniqueness* in (a) and then move on to *existence* in (b)-(d).

(a) Show that equation (1) has *at most* one solution satisfying the initial condition $x(0)=100$. *Hint: Assume that you have two solutions x_1, x_2 to (1). Define a new function $y := x_1 - x_2$ and show that this satisfies $y'(t)+y(t)=0$ with initial condition $y(0)=0$. Show that $y(t)=0$ for all t.*

(b) Part (a) says that if we *guess* a solution of (1) that satisfies $x(0)=100$, then this guess is the only correct solution. In differential equations an educated guess is called an *ansatz* (in German this means 'approach' or 'attempt'). Find values of A, B for which the ansatz

$$x(t)=e^t(A\sin(t)+B\cos(t))$$

satisfies the differential equation in (1) (at this point don't worry if this doesn't satisfy the initial condition $x(0) = 100$, because it won't). *Hint: Plug* $x(t) = e^t(A\sin(t)+B\cos(t))$ *into the differential equation and see what* A, B *have to be, then check that your answer works.*

Your solution to Part (b) does not satisfy the initial condition $x(0) = 100$. Now we will rectify this.

(c) Find all solutions to the differential equation

$$y'(t) + y(t) = 0. \tag{2}$$

Note that you should have a 1-parameter (1-dimensional) family of solutions.

(d) Suppose x_1 is a solution to (1) and y_2 is a solution to (2). Show that $x_1 + y_2$ is a solution of (1). Then choose y_2 appropriately so that $x := x_1 + y_2$ satisfies $x(0) = 100$ as well.

3. Let $b, c \in \mathbb{R}$ be constants, and consider the differential equation

$$x''(t) + bx'(t) + cx(t) = 0. \tag{3}$$

In this problem we will find all solutions to this equation.

(a) Show that if x_1 and x_2 are solutions, then so is $Ax_1 + Bx_2$ for any constants A, B. (This is the mathematical formulation of the *principle of superposition* in physics.)

(b) To guess a solution to (3), begin with the ansatz that $x(t) = e^{rt}$, where $r \in \mathbb{R}$ is a constant. Plug this ansatz into (3) and find all values of r for which e^{rt} is a solution (these values of r will depend on b, c).

By parts (a) and (b) you should have found that all solutions of (3) are of the form $Ae^{r_1 t} + Be^{r_2 t}$ for some constants r_1, r_2 depending on b, c. It turns out these are the only solutions to (3) when $b^2 - 4c \neq 0$ (you will see a proof of this if you take a differential equations class, but for now just assume it is true). Now we will examine some interesting cases of this equation.

(c) Find the solution of the following differential equation with initial conditions:

$$x''(t) - 4x'(t) + 3x(t) = 0, \qquad x(0) = 1, \qquad x'(0) = 0.$$

(d) Use parts (a) and (b) to solve (3) with $b = 0$ and $c = -1$, and with initial conditions $x(0) = 0$ and $x'(0) = 1$. What does this tell you about $\sinh(t)$?

(e) Use parts (a) and (b) to solve (3) with $b = 0$ and $c = 1$, and with initial conditions $x(0) = 0$ and $x'(0) = 1$. What does this tell you about $\sin(t)$?

(f) Let $b, c, d \in \mathbb{R}$ be constants, and consider the differential equation

$$x'''(t) + bx''(t) + cx'(t) + dx(t) = 0. \tag{4}$$

Come up with an ansatz for this differential equation and describe how you would solve it in general (you do not need to actually solve for r in terms of b, c, d). Use your strategy to find all solutions to (4) when $b = c = 0$ and $d = -1$.