1 The Definite Integral As Area

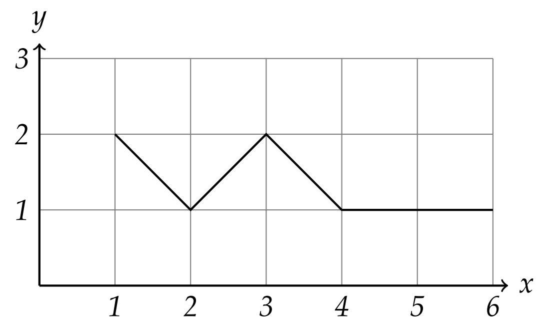
*** The Definite Integral as an Area: When $f(x)$ is Positive**

When $f(x)$ is positive and $a < b$:

$$\text{Area under graph of } f \text{ between } a \text{ and } b = \int_a^b f(x)dx.$$

Example 1 *Find the area under the graph of $y = x^3 + 2$ between $x = 0$ and $x = 2$.*

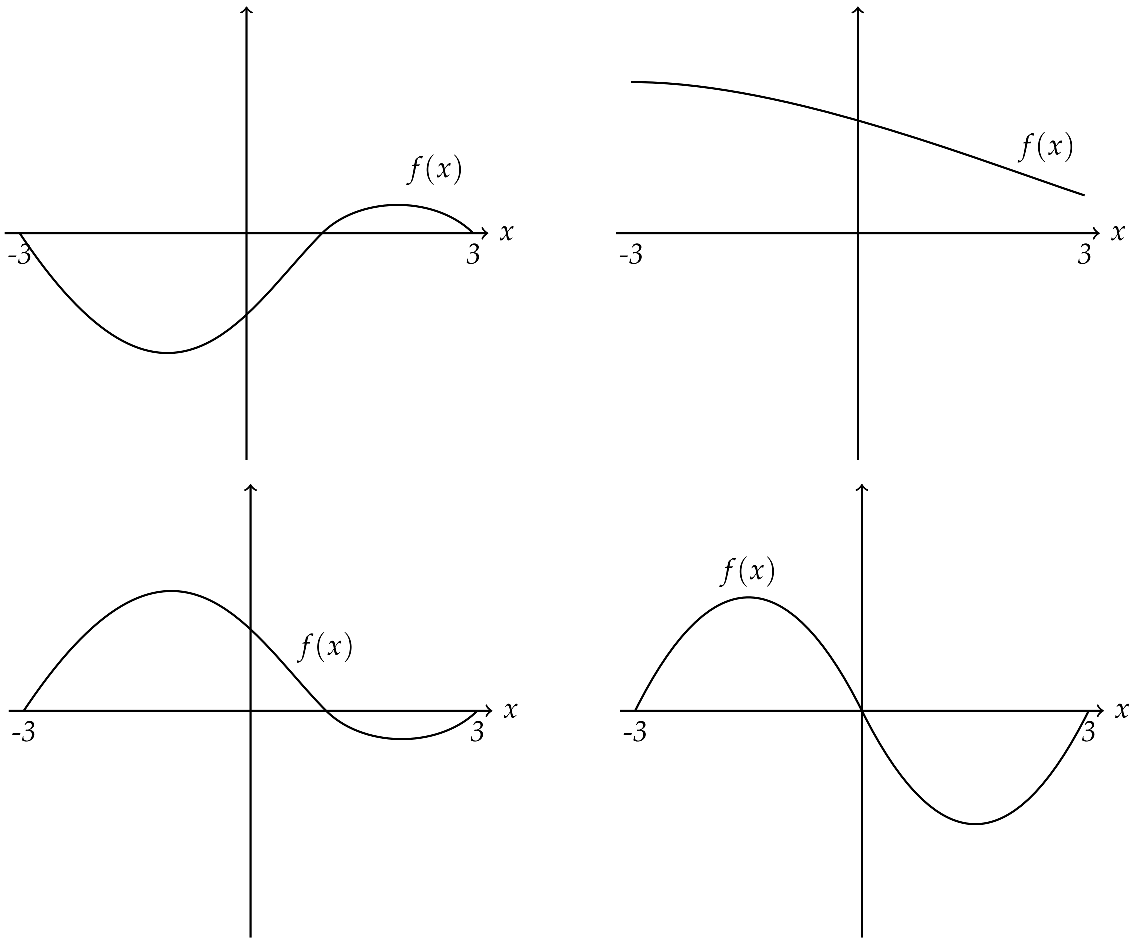
Example 2 *Using the following graph of $y = f(x)$, find the value of $\int_1^6 f(x)dx$.*



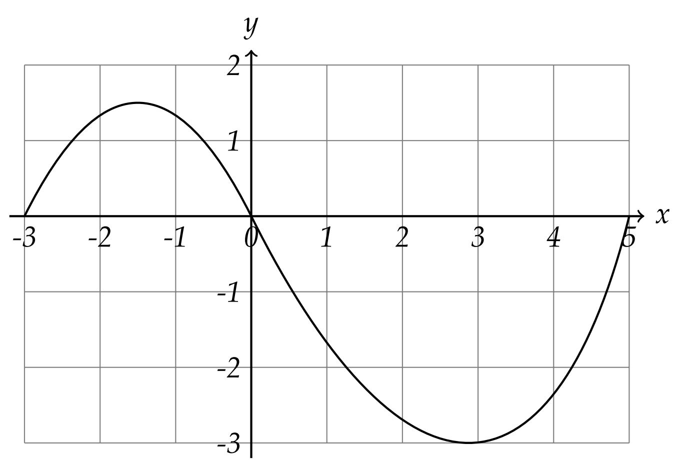
*** Relationship Between Definite Integral and Area: When $f(x)$ is Not Positive**

> **When $f(x)$ is positive for some x-values and negative for others**, and $a < b$:
> $\int_a^b f(x)dx$ is the sum of the areas above the x-axis, counted positively, and the areas below the x-axis, counted negatively.

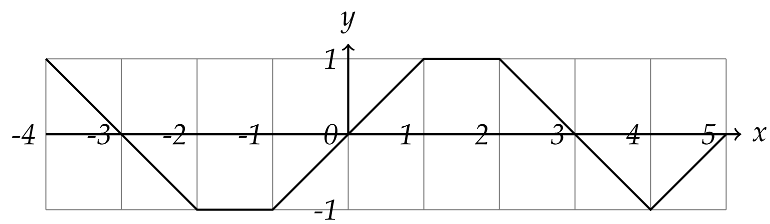
Example 3 *For each of the function $f(x)$ graphed below, decide whether $\int_{-3}^{3} f(x)dx$ is positive, negative or approximately zero.*



Example 4 *Use the following graph of $y = f(x)$ to estimate $\int_{-3}^{5} f(x)dx$*



Example 5 *Given the graph of $y = f(x)$ in the below.*



(a) *Find* $\int_{-3}^{0} f(x)dx$.

(b) *Find* $\int_{0}^{2} f(x)dx$.

(c) *Find* $\int_{1}^{5} f(x)dx$.

(d) *Find* $\int_{-3}^{3} f(x)dx$.

Example 6 *Use the following table to estimate the area between $f(x)$ and the x-axis on the interval* $0 \leq x \leq 20$.

x	0	5	10	15	20
$f(x)$	15	18	20	16	12

*** Area Between Two Curves**

If $g(x) \leq f(x)$ for $a \leq x \leq b$, then

Area between graphs of $f(x)$and $g(x)$for $a \leq x \leq b = \int_a^b (f(x) - g(x))dx.$

Alternatively, without the condition $g(x) \leq f(x)$,

Area between graphs of $f(x)$and $g(x)$for $a \leq x \leq b = \int_a^b |f(x) - g(x)|dx.$

Example 7 *Use an definite integral to find the area under* $y = 5\ln(2x)$ *and above* $y = 3$ *for* $3 \leq x \leq 5$.

Example 8 *Find the area between* $y = x + 5$ *and* $y = 2x + 1$ *between* $x = 0$ *and* $x = 2$.

Example 9 *Use a definite integral to find the area enclosed by* $y = 2 + 8x - 3x^2$ *and* $y = -22 + 2x$.

2 The Fundamental Theorem of Calculus

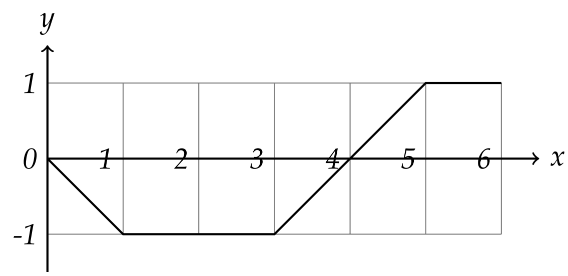
*** The Fundamental Theorem of Calculus**

The Fundamental Theorem of Calculus
If $F'(t)$ is continuous for $a \leq t \leq b$, then

$$\int_a^b F'(t)dt = F(b) - F(a).$$

In words: The definite integral of the derivative of a function gives the total change in the function.

Example 1 *The graph of a derivative $f'(x)$ is shown in the following figure.*



Fill in the table of values for $f(x)$ given that $f(3) = 2$.

x	0	1	2	3	4	5	6
$f(x)$				2			

*** Marginal Cost and Change in Total Cost**

If $C'(q)$ is a marginal cost function and $C(0)$ is the fixed cost,

$$\begin{aligned}
\text{Cost to increase production from } a \text{ units to } b \text{ units} &= C(b) - C(a) = \int_a^b C'(q)dq \\
\text{Total variable cost to produce } b \text{ units} &= \int_0^b C'(q)dq \\
\text{Total cost of producing } b \text{ units} &= \text{Fixed cost } + \text{Total variable cost} \\
&= C(0) + \int_0^b C'(q)dq
\end{aligned}$$

Example 2 *The total cost in dollars to produce q units of a product is $C(q)$. Fixed costs are \$20,000. The marginal cost is*

$$C'(q) = 0.005q^2 - q + 56.$$

(a) On a graph of $C'(q)$, illustrate graphically the total variable cost of producing 150 units.

(b) Estimate $C(150)$, the total cost to produce 150 units.

(c) Find the value of $C'(150)$ and interpret your answer in terms of costs of production.

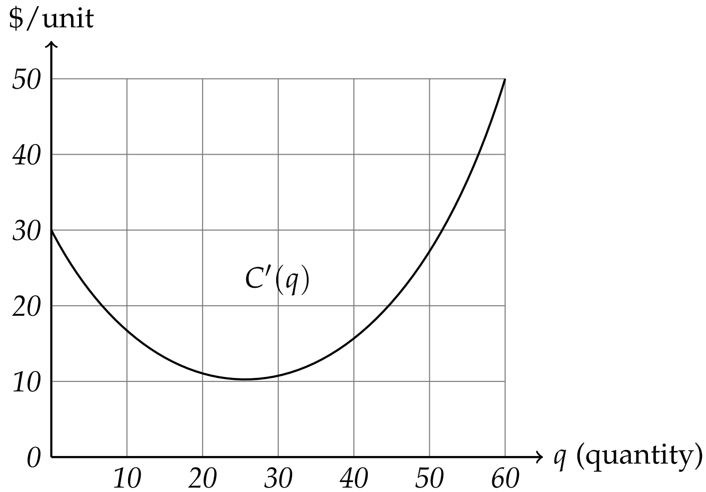
(d) Use parts (b) and (c) to estimate $C(151)$.

Example 3 *A marginal cost function $C'(q)$ is given in the following figure. If the fixed costs are \$10,000, estimate:*

(a) *The total cost to produce 30 units.*

(b) *The additional cost if the company increases production from 30 units to 40 units.*

(c) *The value of $C'(25)$. Interpret your answer in terms of costs of production.*



Example 4 *The marginal cost $C'(q)$ (in dollars per unit) of producing q units is given in the following table.*

q	0	100	200	300	400	500	600
$C'(q)$	25	20	18	22	28	35	45

(a) If fixed cost is \$10,000, estimate the total cost of producing 400 units.

(b) How much would the total cost increase if production were increased one unit, to 401 units?

Example 5 *The marginal cost function of producing q mountain bikes is*

$$C'(q) = \frac{600}{0.3q + 5}.$$

(a) If the fixed cost in producing the bicycle is \$2000, find the total cost to produce 30 bicycles.

(b) If the bikes are sold for \$200 each, what is the profit (or loss) on the first 30 bicycles?

(c) Find the marginal profit on the 31st bicycle.

3 Interpretations of the Definite Integral

*** The Notation and Units for the Definite Integral**

The unit of measurement for

$$\int_a^b f(x)dx$$

is the product of the units for $f(x)$ and the units for x.

If $f(t)$ is a rate of change of a quantity, then the **Total change in quantity between** $t = a$ **and** $t = b$ is given by

$$\int_a^b f(t)dt.$$

Example 1 *A bacteria colony initially has a population of 14 million bacteria. Suppose that t hours later the population is growing at a rate of* $f(t) = 2^t$ *million bacteria per hour.*

(a) Give a definite integral that represents the total change in the bacteria population during the time from $t = 0$ *to* $t = 2$.

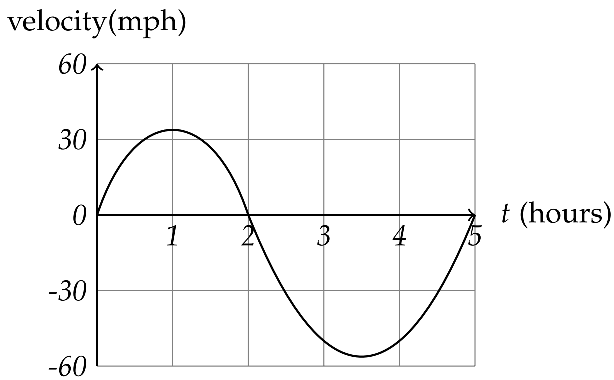
(b) Find the population at time $t = 2$.

Example 2 *Suppose that* $C(t)$ *represents the cost per day to heat your home in dollars per day, where t is time measured in days and* $t = 0$ *corresponds to January 1, 2010. Interpret* $\int_0^{90} C(t)dt$.

Example 3 *Interpret* $\int_1^3 v(t)dt$*, where* $v(t)$ *is velocity in meters/sec and t is time in seconds.*

Example 4 *A cup of coffee at 90° is put into a 20° room when $t = 0$. The coffees's temperature is changing at a rate of $r(t) = -7(0.9^t)^\circ$ per minute, with t in minutes. Estimate the coffee's temperature when $t = 10$.*

Example 5 *A man starts 50 miles away from his home and takes a trip in his car. He moves on a straight line, and his home lies on this line. His velocity is given in the following figure and positive velocities take him toward home.*



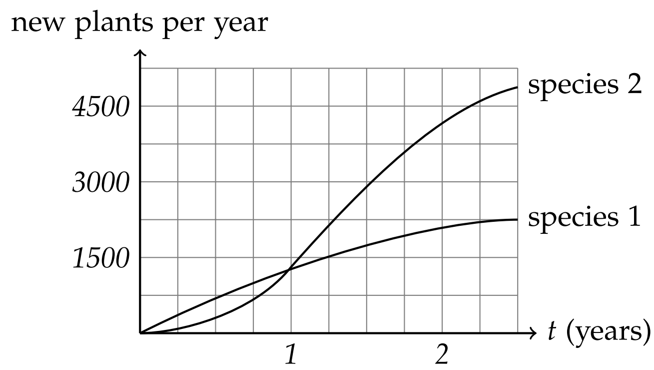
(a) *Does the man turn around? If so, at what time(s)?*

(b) *When is he going the fastest? How fast is he going then? Toward his home or away?*

(c) *When is he closest to his home? Approximately how far away is he then?*

(d) *When is the man farthest from his home? How far away is he then?*

Example 6 *The rates of growth of the populations of two species of plants are shown in the following figure. Assume that the populations of the two species are equal at time* $t = 0$.



(a) *Which population is larger after one year? After two years?*

(b) *How much does the population of species 1 increase during the first two years?*

Example 7 *The following graph shows the rate of change of the quantity of water in a water tower, in liters per day, during the month of April. If the tower had 12,000 liters of water in it on April 1, estimate the quantity of water in the tower on April 30.*