* Concavity

The graph of a function is concave up if it bends upward as we move left to right, or the AROC is increasing from left to right.
The graph of a function is concave down if it bends downward as we move left to right, or the AROC is decreasing from left to right.
A line is neither concave up nor concave down.

Example 1  The figure below shows the graph of \( y = f(x) \). Estimate the intervals over which:

(a) The function is increasing; decreasing.

(b) The graph is concave up; concave down.

Example 2  The following table gives values of \( f(t) \). Is \( f \) increasing or decreasing? Is the graph of \( f \) concave up or concave down?

<table>
<thead>
<tr>
<th>( t )</th>
<th>( 0 )</th>
<th>( 5 )</th>
<th>( 10 )</th>
<th>( 15 )</th>
<th>( 20 )</th>
<th>( 25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>13.1</td>
<td>14.1</td>
<td>16.2</td>
<td>20.0</td>
<td>29.6</td>
<td>42.7</td>
</tr>
</tbody>
</table>

Example 3  The following table gives values of \( g(t) \). Is \( g \) increasing or decreasing? Is the graph of \( g \) concave up or concave down?

<table>
<thead>
<tr>
<th>( t )</th>
<th>( 0 )</th>
<th>( 2 )</th>
<th>( 4 )</th>
<th>( 6 )</th>
<th>( 8 )</th>
<th>( 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(t) )</td>
<td>40</td>
<td>37</td>
<td>33</td>
<td>28</td>
<td>22</td>
<td>15</td>
</tr>
</tbody>
</table>
1 The Second Derivative

* What Does the Second Derivative Tell Us?

\[ f'' > 0 \] on an interval means \( f' \) is increasing, so the graph of \( f \) is **concave up** there.

\[ f'' < 0 \] on an interval means \( f' \) is decreasing, so the graph of \( f \) is **concave down** there.

**Example 1** Given the graph of the function \( f(x) \) below, determine whether quantities are positive, negative or zero?

Fill in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( f''(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 2 Consider the following graph of \( y = f(x) \).

(a) Estimate the intervals on which the derivative is positive and the intervals on which the derivative is negative.

(b) Estimate the intervals on which the second derivative is positive and the intervals on which the second derivative is negative.

Example 3 Graph the functions described in parts (a)-(d).

(a) First and second derivatives everywhere positive.

(b) Second derivative everywhere negative; first derivative everywhere positive.

(c) Second derivative everywhere positive; first derivative everywhere negative.

(d) First and second derivatives everywhere negative.
Example 4 For each function given in the following tables, do the signs of the first and second derivatives of the function appear to be positive or negative over the given interval?

(a) \[
\begin{array}{c|c|c|c|c|c|c}
 x & 1.0 & 1.1 & 1.2 & 1.3 & 1.4 & 1.5 \\
 f(x) & 10.1 & 11.2 & 13.7 & 16.8 & 21.2 & 27.7 \\
\end{array}
\]

(b) \[
\begin{array}{c|c|c|c|c|c|c|c}
 x & 1.0 & 1.1 & 1.2 & 1.3 & 1.4 & 1.5 \\
 g(x) & 10.1 & 9.9 & 8.1 & 6.0 & 3.5 & 0.1 \\
\end{array}
\]

(c) \[
\begin{array}{c|c|c|c|c|c|c}
 x & 1.0 & 1.1 & 1.2 & 1.3 & 1.4 & 1.5 \\
 h(x) & 1000 & 1010 & 1015 & 1018 & 1020 & 1021 \\
\end{array}
\]

(d) \[
\begin{array}{c|c|c|c|c|c|c|c}
 x & 10 & 20 & 30 & 40 & 50 \\
 w(x) & 10.7 & 6.3 & 4.2 & 3.5 & 3.3 \\
\end{array}
\]

Example 5 Sketch a graph of a continuous function \( f \) with the following properties:

(a) \( f(0) = 1 \)

(b) \( f'(x) > 0 \) for all \( x \)

(c) \( f''(x) < 0 \) for \( x < 0 \)

(d) \( f''(x) > 0 \) for \( x > 0 \)

(e) \( f'(0) = 1 \)

Example 6 Sketch a graph of a function \( f \) such that \( f(1) = 4, f'(1) = 1/2 \) and \( f''(x) > 0 \).

Example 7 A function \( f \) has \( f(3) = 10, f'(3) = 2, \) and \( f''(x) < 0 \) for \( x \geq 3 \). Which of the following are possible values for \( f(7) \)? (a) 20 (b) 18 (c) 16
2 Local Maxima and Minima

* Local Maxima and Minima

Suppose $p$ is a point in the domain of $f$:

- $f$ has a **local minimum** at $p$ if $f(p)$ is less than or equal to the values of $f$ for points near $p$.

- $f$ has a **local maximum** at $p$ if $f(p)$ is greater than or equal to the values of $f$ for points near $p$.

* How Do We Detect a Local Maximum or Minimum?

Suppose $p$ is a point in the domain of $f$:

For any function $f$, a point $p$ in the domain of $f$ where $f'(p) = 0$ or $f'(p)$ is undefined is called a **critical point** of the function.

If a function, continuous on an interval (its domain), has a local maximum or minimum at $p$, then $p$ is a critical point or an endpoint of the interval.

**Example 1** For $f(x)$ given below, indicate all critical points of the function $f$. How many critical points are there? Identify each critical point as a local maximum, a local minimum, or neither.

![Graphs showing local maxima and minima](image-url)
Example 2  (a) Graph a function with two local minima and one local maximum.

(b) Graph a function with two critical points. One of these critical points should be a local minimum, and the other should be neither a local maximum nor a local minimum.

* Testing For Local Maxima and Minima

First Derivative Test for Local Maxima and Minima
Suppose \( p \) is a critical point of a continuous function \( f \). Then, as we go from left to right:

- If \( f \) changes from decreasing to increasing at \( p \), then \( f \) has a local minimum at \( p \).
- If \( f \) changes from increasing to decreasing at \( p \), then \( f \) has a local maximum at \( p \).

Example 3  (a) Graph a function \( f \) with the following properties:

- \( f(x) \) has critical points at \( x = 2 \) and \( x = 5 \);
- \( f'(x) \) is positive to the left of 2 and positive to the right of 5;
- \( f'(x) \) is negative between 2 and 5.

(b) Identify the critical points as local maxima, local minima, or neither.
Example 4  Given the graph of \( f'(x) \) below.

(a) What are the critical points of the function \( f(x) \)?

(b) Identify each critical point as a local maximum, a local minimum, or neither.

Example 5  Given \( f(x) = x^3(1 - x)^4 \).

(a) Find all critical points of \( f \).

(b) Use the first derivative test to classify each critical point as a local max, a local min, or neither.
Second Derivative Test for Local Maxima and Minima
Suppose $p$ is a critical point of a continuous function $f$, and $f'(p) = 0$.

- If $f$ is concave up at $p$ ($f''(p) > 0$), then $f$ has a local minimum at $p$.
- If $f$ is concave down at $p$ ($f''(p) < 0$), then $f$ has a local maximum at $p$.

Example 6
Given $f(x) = x^3 - 9x^2 - 48x + 52$.

(a) Find all critical points of $f$.

(b) Use the second derivative test to classify each critical point as a local max, a local min, or neither.

Example 7
Find constants $a$ and $b$ so that $f(x) = a(x - b \ln x)$ has a local minimum at the point $(2, 5)$.
3 Inflection Points

* Concavity and Inflection Points

A point at which the graph of a function $f$ changes concavity is called an inflection point of $f$.

If $p$ is an inflection point of $f$, then either $f''(p) = 0$ or $f''$ is undefined at $p$.

Example 1 For the graph of $f(x)$ given below, indicate the approximate locations of all inflection points. How many inflection points are there?
Example 2 Find the inflection points of $f(x) = x^3 - 9x^2 - 48x + 52$.

Example 3 Find the critical points and inflection points of $f(x) = xe^{-x}$.

Example 4 Graph a function $f$ with the following properties: $f$ has a critical point at $x = 4$ and an inflection point at $x = 8$; the value of $f'$ is negative to the left of 4 and positive to the right of 4; the value of $f''$ is positive to the left of 8 and negative to the right of 8.
Example 5 Sketch a graph of $y = f(x)$ such that

(a) $f'(-1) = 0$, $f'(3) = 0$

(b) $f'(x) > 0$ for $x < -1$ and $-1 < x < 3$

(c) $f'(x) < 0$ for $x > 3$

(d) $f''(-1) = 0$, $f''(1) = 0$

(e) $f''(x) < 0$ for $x < -1$ and $x > 1$

(f) $f''(x) > 0$ for $-1 < x < 1$