The Natural Logarithm

*** Definition and Properties of the Natural Logarithm**

The **natural logarithm** of x, written $\ln x$, is the power of e needed to get x. In other words,

$$\ln x = c \qquad \text{means} \qquad e^c = x.$$

The natural logarithm is sometimes written $\log_e^x$.
$\ln x$ is not defined if x is negative or 0.

Properties of the Natural logarithm

$$\begin{aligned} \ln(AB) &= \ln A + \ln B \qquad \text{(Product Rule)} \\ \ln\left(\frac{A}{B}\right) &= \ln A - \ln B \qquad \text{(Quotient Rule)} \\ \ln(A^p) &= p \ln A \qquad \text{(Power Rule)} \\ \ln e^x &= x \\ e^{\ln x} &= x \end{aligned}$$

In addition, $\ln 1 = 0$ and $\ln e = 1$.

*** Solving Equations Using Logarithms**

Example 1 *Solve* $130 = 2^t$ *for t using natural logarithms.*

Example 2 *Solve* $100 = 25\,(1.5)^t$ *for t using natural logarithms.*

Example 3 *Solve* $5 = 2\,e^t$ *for* t *using natural logarithms.*

Example 4 *Solve* $5e^{3t} = 8e^{2t}$ *for* t *using natural logarithms.*

Example 5 *Solve* $7 \cdot 3^t = 5 \cdot 2^t$ *for* t *using natural logarithms.*

*** Exponential Functions with Base** e

Writing $a = e^k$, so $k = \ln a$, any exponential function can be written in two forms

$$P = P_0 a^t \qquad \text{or} \qquad P = P_0 e^{kt}.$$

If $a > 1$, we have exponential growth; if $0 < a < 1$, we have exponential decay.
If $k > 0$, we have exponential growth; if $k < 0$, we have exponential decay.
k is called the continuous growth or decay rate.

Example 6 *A town's population is 2000 and growing at 5% a year.*

(a) *Find a formula for the population at time t years from now assuming that 5% per year is an annal rate.*

(b) *Find a formula for the population at time t years from now assuming that 5% per year is a continuous annual rate.*

Example 7 (a) *Convert the function* $P = 20\, e^{-0.5t}$ *to the form* $P = P_0 a^t$.

(b) *Convert the function* $P = P_0\, e^{0.2t}$ *to the form* $P = P_0 a^t$.

(c) *Convert the function* $P = 10\,(1.7)^t$ *to the form* $P = P_0 e^{kt}$.

(d) *Convert the function* $P = 4\,(0.55)^t$ *to the form* $P = P_0 e^{kt}$.

Which of the functions above represents exponential growth and which represents exponential decay?

***Exponential Growth and Decay**

Many quantities in nature change according to an exponential growth or decay function of the form

$$P = P_0 e^{kt},$$

where P_0 is the initial quantity and k is the continuous growth or decay rate.

Example 8 *In 1990, the population of Africa was 643 million and by 2000 it had grown to 819 million.*

(a) *Assuming the population increases exponentially at a continuous rate, find a formula for the population of Africa as a function of time t in years since 1990.*

(b) *By which year will Africa's population reach 2000 million?*

*** Doubling Time and Half-Life**

The **doubling time** of an exponentially increasing quantity is the time required for the quantity to double.
The **half-life** of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.

Example 9 *Find the doubling time of a quantity that is exponentially increasing by an annual rate of* 10% *per year.*

Example 10 *The quantity of ozone, Q, is decaying exponentially at a continuous rate of* 0.25% *per year. What is the half-life of ozone?*

Example 11 *Strontium-90 is a waste product from nuclear reactors, which decays exponentially The half-life of strontium-90 is 29 years. Estimate the percent of original strontium-90 remaining after 100 years?*

*** Financial Applications: Compound Interest**

An amount P_0 is deposited in an account paying interest at a rate of r per year. Here r is the decimal representation of the percentage. Let P_0 be the initial deposit. Let P be the balance in the account after t years.
If interest is **compounded annually**, then $P = P_0\,(1+r)^t$.
If interest is **compounded continuously**, then $P = P_0\,e^{rt}$, where e is the natural base.

Example 12 *If you deposit* \$3000 *in an account earning interest at an* 8% *annual rate. How much is in the account after 10 years if the interest is compounded*

(a) Annually?

(b) Continuously?

Example 13 *If* 10,000 *is deposited in an account paying* 10% *interest per year, compounded continuously, how long will it take for the balance to reach* 25,000*?*

*** Present and Future Value**

The **future value**, B, of a payment, P, is the amount to which the P would have grown if deposited today in an interest-bearing bank account.
The **present value**, P, of a future payment, B, is the amount that would have to be deposited in a bank account today to produce exactly B in the account at the relevant time in future.

Suppose B is the *future value* of P and P is the *present value* of B.
If interest is compounded annually at a rate r for t years, then

$$B = P(1+r)^t, \qquad \text{or equivalently,} \qquad P = \frac{B}{(1+r)^t}.$$

If interest is compounded continuously at a rate r for t years, then

$$B = P e^{rt}, \qquad \text{or equivalently,} \qquad P = \frac{B}{e^{rt}} = B e^{-rt}.$$

Example 14 *Find the future value in 8 years of a* \$10,000 *payment today, if the interest rate is* 3% *per year compounded continuously.*

Example 15 *Find the present value of an* \$8000 *payment to be made in 5 years. The interest rate is* 4% *per year compounded annually.*