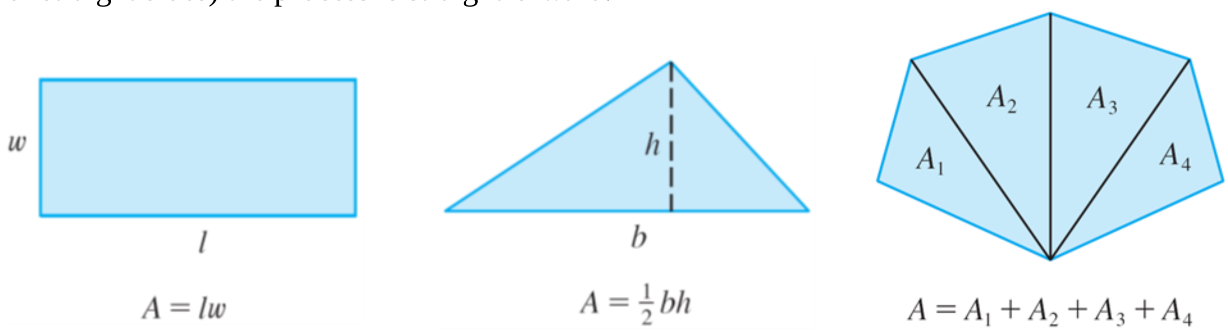
Section 4.1 – Area, Distance, and Sigma Notation

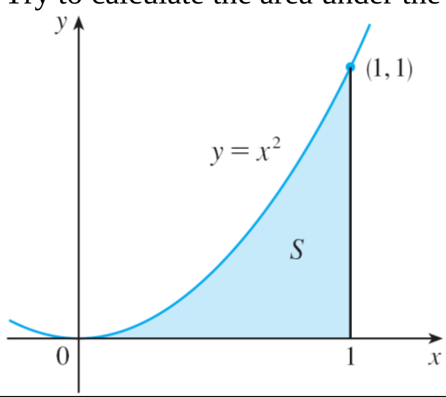
The Area Problem

Our goal is to calculate the area of some region. If that region is a polygon (a shape made only of straight sides) the process is straightforward.

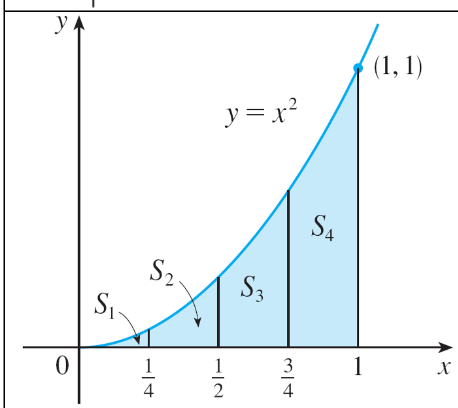


But what if that region has curved edges?

Try to calculate the area under the curve $y = x^2$ from 0 to 1.

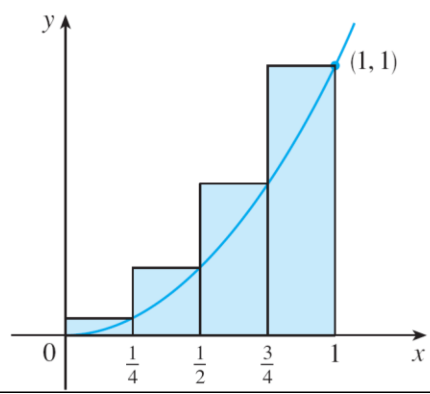


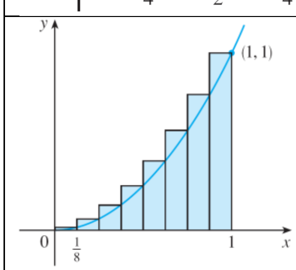
The curved edge means we cannot divide the region into triangles like we would with a large polygon. Instead of trying to calculate the area exactly, lets estimate the area by splitting the region into smaller regions



Now each smaller region can be approximated with a rectangle of width 1/4.

On the right we use an over approximation.



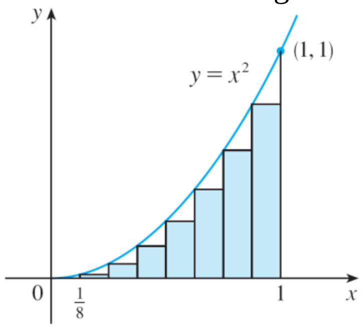


(b) Using right endpoints

If we use more rectangles, our approximation can be more accurate.

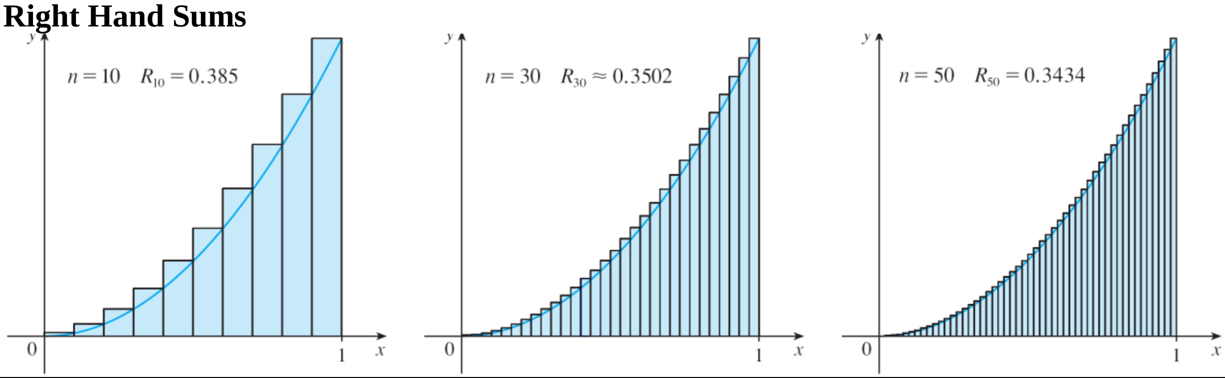
If we use n rectangles, then each has a width of $1/n$

Above we looked at **Right Hand Sums**, meaning we used the right side of each rectangle for our approximation. A **Left Hand Sum** is the same approximation process, except we use the left side of the rectangle.

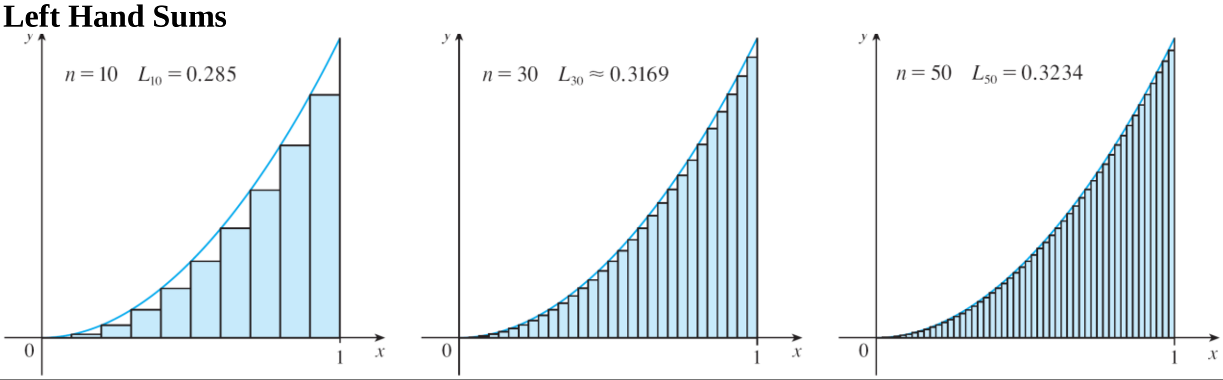


(a) Using left endpoints

Right Hand Sums



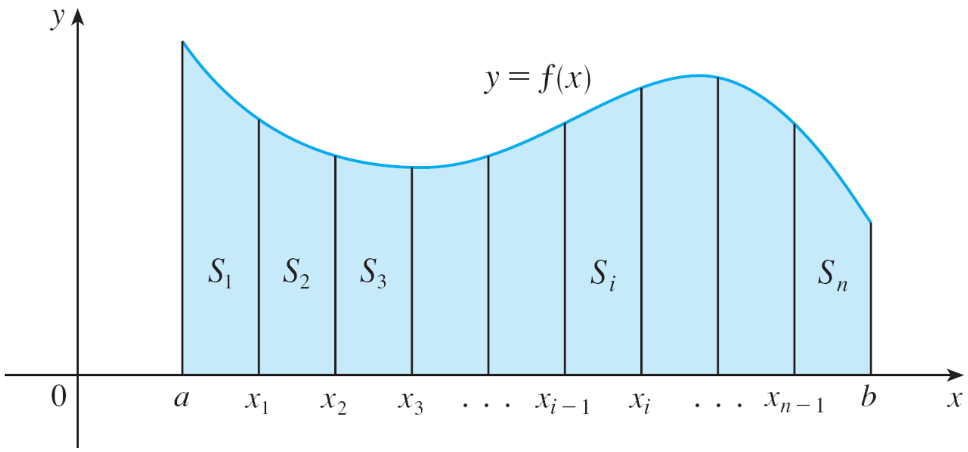
Left Hand Sums



If n is the number of rectangles, R_n is the right hand sum with n rectangles, and L_n is the left hand sum with n rectangles, then

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \frac{1}{3}$$

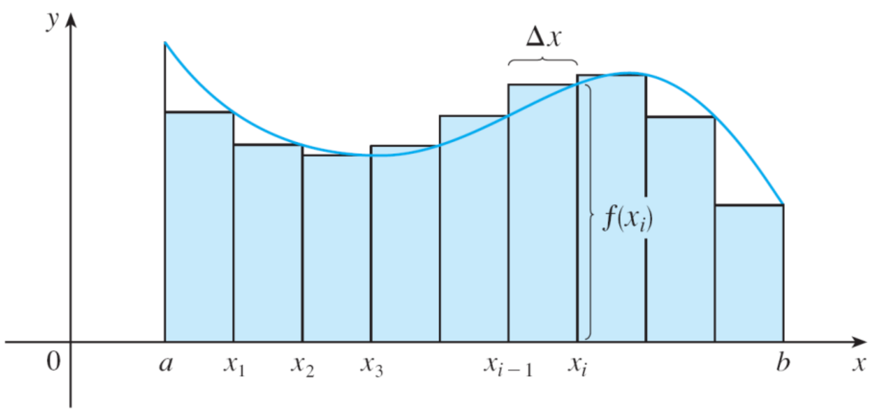
Let's apply what we learned to a general region from *a* to *b*



Here we have *n* regions of equal width that we will approximate with rectangles. The first thing we need to find is the width of each rectangle. The width of $[a, b]$ is $b - a$, so the width of each strip (call it Δx (*delta x*)) is

$$\Delta x = \frac{b-a}{n}$$

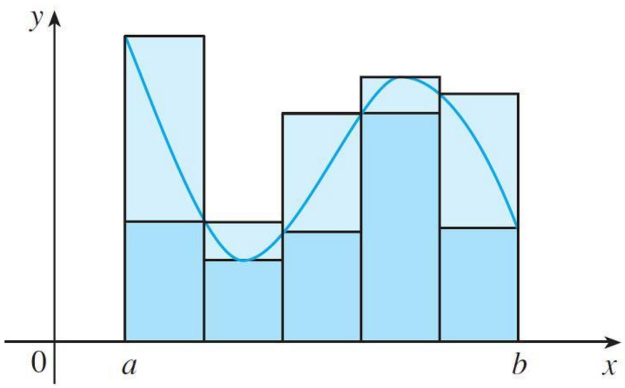
So our approximation looks like



The height of each rectangle is simply the function value on one side, $f(x_i)$. So the area

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

Finally, an **Upper Sum** is a sum that uses the largest function value in each interval rather than an endpoint. A **Lower Sum** uses the smallest value in each interval.



If a function is **strictly increasing**, then an **Upper Sum** is a **Right Hand Sum** and a **Lower Sum** is a **Left Hand Sum**.

Example 1: Estimate the area under the graph of $f(x) = x^2 + 2x$ from 4 to 16 using the areas of 3 rectangles of equal width, with heights of the rectangles determined by the height of the curve at both **left endpoints** and **right endpoints**.

Example 2: Using the function $f(x) = 63 + 2x - x^2$ from -7 to 9, overestimate the area using an **Upper Sum** and underestimate the area using a **Lower Sum**. Use 4 rectangles.

Example 3: The following table gives the velocity (in m/s) of an object at time *t* (in seconds).

t	2	4	6	8	10
v(t)	40	38	32	25	10

Estimate the distance traveled using a left and a right hand sum.

Example 4: Find the sum of all numbers from 1 to 100

Sigma Notation $$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5$$
Summation Rules $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$ $$\sum_{i=1}^{n} (ca_i) = c\sum_{i=1}^{n} a_i$$ $$\sum_{i=1}^{n} 1 = n$$ $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$ $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$ $$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Example 5: Find the following sums

$$\sum_{i=1}^{100} 2i =$$

$$\sum_{i=31}^{100} 2i =$$

Section 4.2 – Definite Integral

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Definite Integrals

If f is integrable on $[a, b]$,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Where $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$

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Definitions

$\int$ *is* called an **integral sign** and looks like an elongated *S*

The function $f(x)$ is called the **integrand**

The **limits of integration** are *a* and *b*

a is the **lower limit**

b is the **upper limit**

dx simply indicates that *x* is the independent variable

The whole procedure is called **integration**

The sum is called a **Riemann sum**

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Example 1: Write the following limit of a Riemann sum as an integral with limits 3 and 10

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{3 + \frac{7i}{n}} \cdot \frac{7}{n}$$

Example 2: Write the following limit of a Riemann sum as an integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{8 + \frac{5i}{n}} \cdot \frac{5}{n}$$

Example 3: Write the following limit of a Riemann sum as an integral

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{24\cdot\frac{3i}{n}+9}{n}$$

What is an integral and when are we allowed to do it?
Integration can be thought of as derivatives in reverse. Where a derivative is the slope of the tangent line at any point on a graph, a definite integral is the area under the curve between two points. So a definite integral exists on a closed interval $[a,b]$ if *f* is continuous on that interval, or if *f* has a finite number of jump discontinuities.

Example 4: Evaluate the following integral by interpreting it in terms of area

$$\int_{-5}^{5}\sqrt{25-x^2}dx$$

Example 5: Evaluate the following integral by interpreting it in terms of area

$$\int_{0}^{10}|x-3|dx$$

Example 6: Evaluate the following integral by interpreting it in terms of area

$$\int_{2}^{8} g(x)dx \qquad g(x) = \begin{cases} 2 & if\ 2 \le x \le 6 \\ 5 & if\ 6 < x \le 8 \end{cases}$$

Integral Properties

1. $\int_a^b c\, dx = c(b-a)$ where c is any constant
2. $\int_a^b x\, dx = \frac{1}{2}(b^2 - a^2)$
3. $\int_a^b x^2\, dx = \frac{1}{3}(b^3 - a^3)$
4. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$
5. $\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$ where c is any constant
6. $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$
7. $\int_a^c f(x)\, dx + \int_c^b f(x)\, dx = \int_a^b f(x)\, dx$
8. If $f(x) \ge 0$ for $a \le x \le b$, then $\int_a^b f(x)\, dx \ge 0$
9. If $f(x) \ge g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\, dx \ge \int_a^b g(x)\, dx$
10. If $m \le f(x) \le M$ for $a \le x \le b$, then
$$m(b-a) \le \int_a^b f(x)\, dx \le M(b-a)$$

Example 7: Evaluate the following integral

$$\int_{-1}^{5} 4x + 3\, dx$$

Example 8: Evaluate the following integral

$$\int_{0}^{3} (x+2)^2 dx$$

Example 9: Evaluate the following integrals given

$$\int_{5}^{13} f(x)\, dx = 12 \qquad \int_{5}^{8} f(x)\, dx = 5 \qquad \int_{10}^{13} f(x)\, dx = 4$$

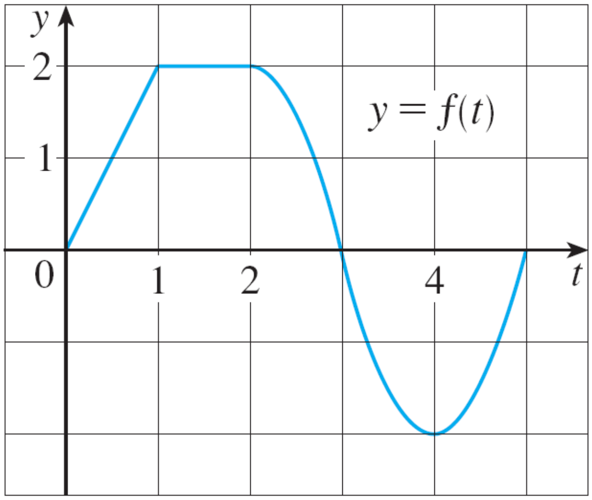
$$\int_{8}^{13} f(x)\, dx =$$

$$\int_{8}^{10} f(x)\, dx =$$

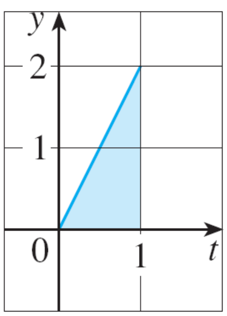
$$\int_{10}^{5} f(x)\, dx =$$

Section 4.3 – Fundamental Theorem

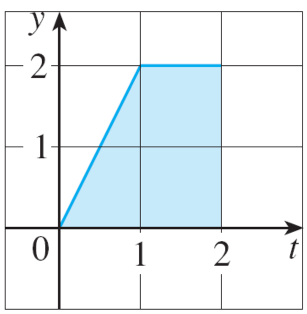
Example 1: Given that *f* is the function below and $g(x) = \int_0^x f(t)dt$, find $g(0), g(1), g(2), g(3), g(4), and\ g(5)$ and then sketch $g(x)$.



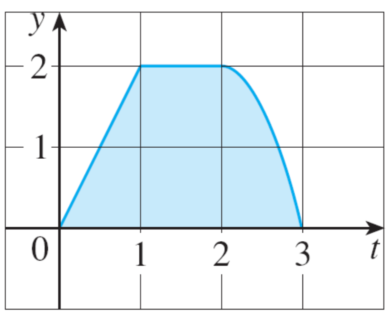
Solution: First, $g(0) = \int_0^0 f(t)dt = 0$. Next we obtain the following



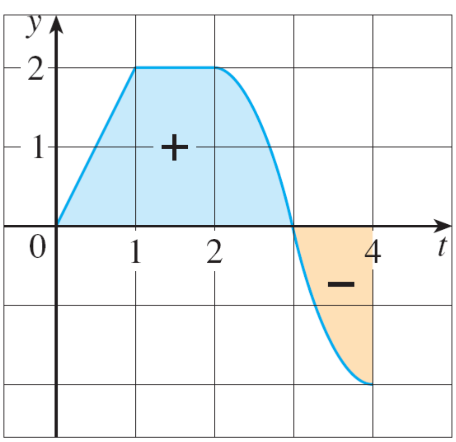
$g(1) = 1$



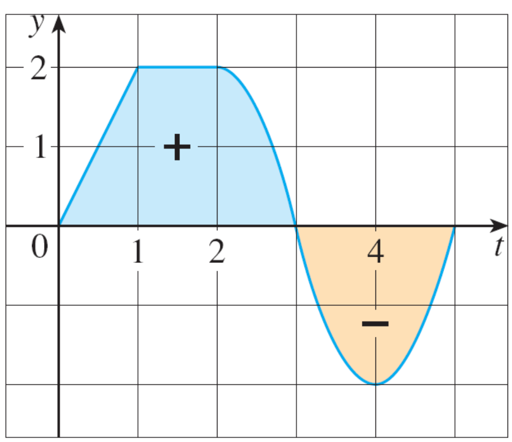
$g(2) = 3$



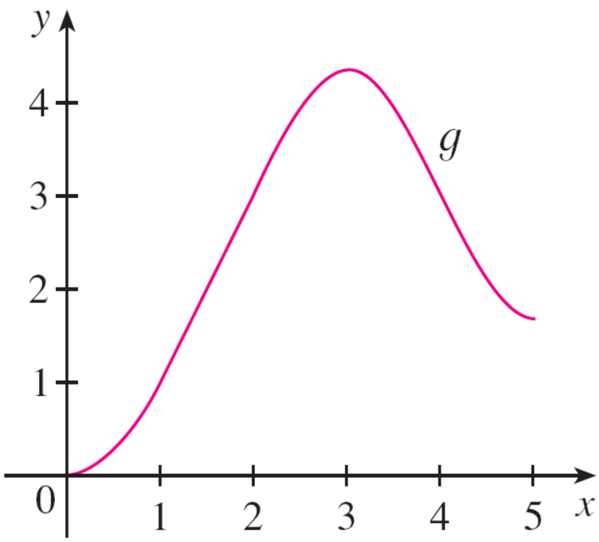
$g(3) \approx 4.3$



$g(4) \approx 3$



$g(5) \approx 1.7$



Using the values found on the previous page, we get sketch on the left for the graph of $g(x)$

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \qquad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$

Written another way we have

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Example 2: Find $g'(x)$ given that

$$g(x) = \int_3^x \frac{1}{1+t^3}dt$$

Example 3: Find $g'(x)$ given that

$$g(x) = \int_{12}^x \sin t \, dt$$

Example 4: Find $g'(x)$ given that

$$g(x) = \int_{-17}^{x} t^2 dt$$

Example 5: Find $g'(x)$ given that

$$g(x) = \int_{x}^{42} \frac{1}{1+t^3} dt$$

The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$, then

$$\int_{a}^{b} f(t)dt = F(b) - F(a)$$

where F is any antiderivative of f

Example 6: Evaluate the integral

$$\int_{-2}^{1} 3x^2 dx$$

Example 7: Evaluate the integral

$$\int_{\pi/6}^{\pi/3} \cos x \, dx$$

Example 8: Find the derivative of the integral below by first using the FTCp2.

$$\frac{d}{dx}\int_{\pi/2}^{x^4} \cos t \, dt$$

Example 9: Find the derivative of the integral below

$$\frac{d}{dx}\int_{3}^{\sin x} e^t dt$$

Example 10: Find the derivative of the integral below

$$\frac{d}{dx}\int_{x^2}^{x^5} \tan t \, dt$$

Example 11: Evaluate the integral

$$\int_{0}^{20} |x - 14| dx$$

Example 12: Use a definite integral to find the area below the curve $y = 12 - 3x^2$ and above the *x*-axis.

Example 13: Find $h'(x)$ given

$$h(x) = \int_{53}^{1+\sin x} \cos t^2 + t \, dt$$

Example 14: Find $g'(x)$ given

$$g(x) = \int_{9x}^{3x} \frac{1}{u^2 + 6} \, du$$

Section 4.4 – Indefinite Integral

Indefinite Integrals

$$\int f(x)dx = F(x) \quad means \quad F'(x) = f(x)$$

For example,

$$\int x^2 dx = \frac{x^3}{3} + C \quad because \quad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

The difference between definite and indefinite integrals is that a definite integral (the ones with specific limits *a* and *b*) is a **number** where an indefinite integral is a **function**.

Table of Common Indefinite Integrals

$\int cf(x)dx = c\int f(x)dx$	$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
$\int k\,dx = kx + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
$\int \cos x\,dx = \sin x + C$	$\int \sin x\,dx = -\cos x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$

Example 1: Find the (most) general indefinite integral

$$\int (x^{41} + x^{52} + 7x^5)dx$$

Example 2: Find the general indefinite integral

$$\int \left(\sqrt[3]{x^2} + \frac{1}{x^5} - \sin x\right)dx$$

Example 3: Find the general indefinite integral

$$\int \frac{\sin\theta}{\cos^2\theta}\,d\theta$$

Example 4: Find the general indefinite integral

$$\int t^4(t-1)dt$$

Example 5: Find the general indefinite integral

$$\int \frac{3x^5-2}{17\sqrt{x}}\,dx$$

Section 5.5 – Total Area

Average Value: the average value of a function f on the interval $[a, b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

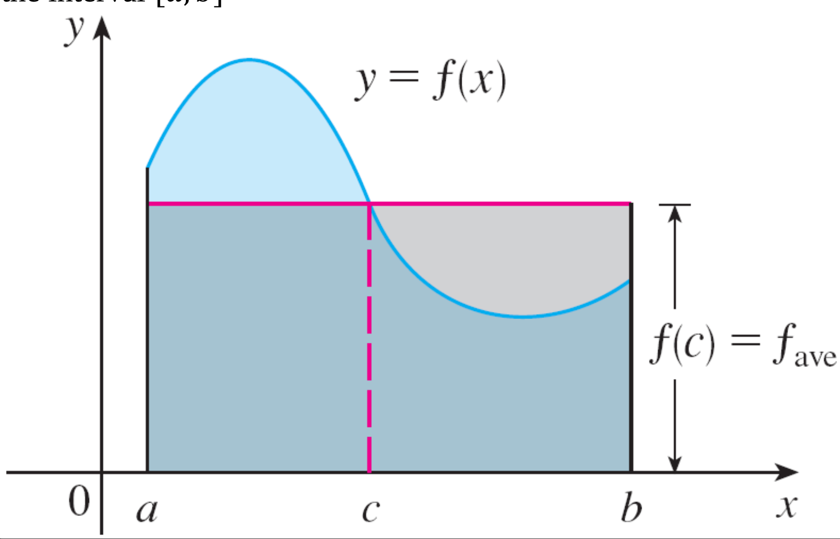
The Mean Value Theorem for Integrals
If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

or rewritten,

$$\int_a^b f(x)dx = f(c)(b-a)$$

Graphically, this means that there is a rectangle with the exact same area and width as the definite integral from a to b, and the height of this rectangle is a function value somewhere in the interval $[a, b]$



Example 6: Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[0, 6]$.

Example 7: MSU *vs* UofM football game. The game clock reads 5:04 when Michigan snaps the ball from their own 27 yard line. A Michigan State cornerback runs down the sideline chasing the Michigan receiver. 6 seconds into the play he makes an interception and starts running back the other way, still straight down the sideline. He is forced out of bounds and the clock reads 4:55. The MSU cornerback's velocity during the play is $v(t) = t(6 - t)$ measured in yd/sec.

a) Where was the corner forced out of bounds?

b) How far did he travel over the course of the play?

c) What was his average velocity?

d) What was his average speed?

Section 4.5 – Substitution

Example 1: Find the derivative of $F(x) = \sin(x^8)$

Now evaluate $F(x) = \int \cos(x^8) \cdot 8x^7 dx$ where $F(0) = 0$

Substitution Rule (often called *u-substitution*)

If $u = g(x)$ is a differentiable function whose range is an interval *I* and f is continuous on *I*, then

$$\int f\big(g(x)\big)g'(x)dx = \int f(u)du$$

Example 2: Use an appropriate substitution to evaluate

$$\int \sec^2(x^4 + 2)\, x^3 dx$$

Example 3: Use a *u* substitution to evaluate

$$\int x\cos x^2 \sin x^2\, dx$$

Example 4: Use a *u* substitution to evaluate

$$\int \frac{x^2 dx}{(3-x^3)^2}$$

Example 5: Use a *u* substitution to evaluate

$$\int \frac{\csc\sqrt{x}\cot\sqrt{x}}{\sqrt{x}}\, dx$$

Substitution Rule for **Definite** Integrals

If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 6: Evaluate the following definite integral

$$\int_{\pi/6}^{2\pi/3} \sin x \cos x \, dx$$

Integrals of **Symmetric** Functions

Suppose f is continuous on $[-a,a]$,

1. If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$
2. If f is odd, then $\int_{-a}^{a} f(x)dx = 0$

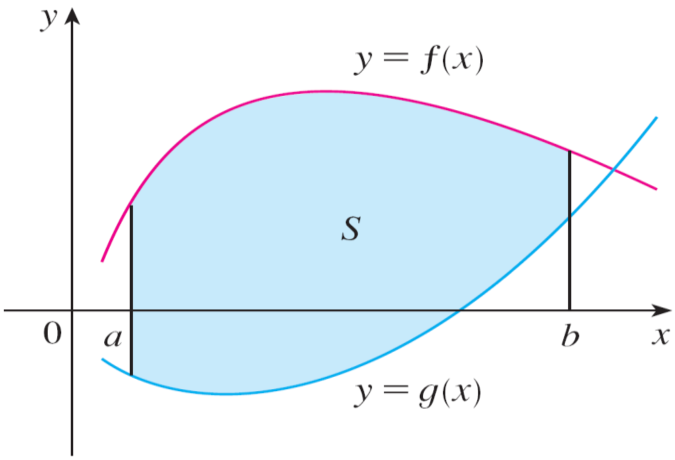
Example 7: Evaluate the following integral without doing any difficult math.

$$\int_{-1098771}^{1098771} \sin x + x^3 - 2x \, dx$$

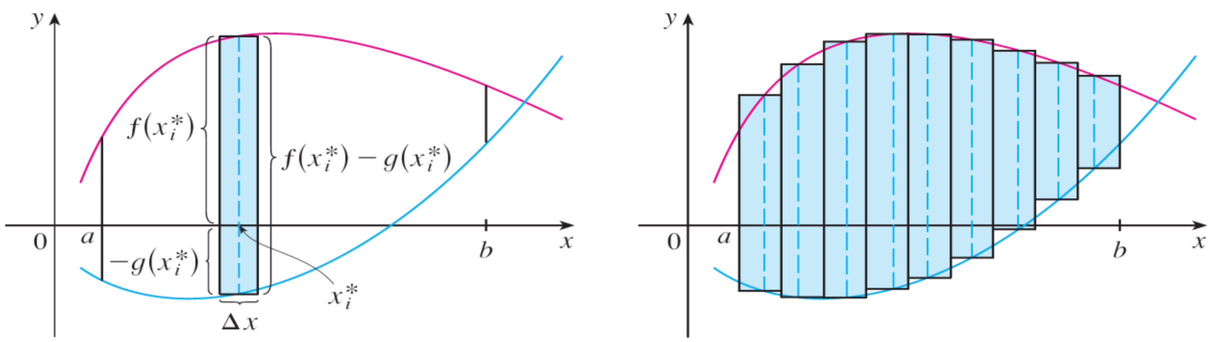
Section 5.1 – Area Between Curves

Area Between Curves

Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$.



Using the same methods for finding the area under a curve we divide the region into n strips and find the area of n rectangles. The main difference this time is that the bottom of the rectangle is not the x-axis, but rather the function $y = g(x)$.

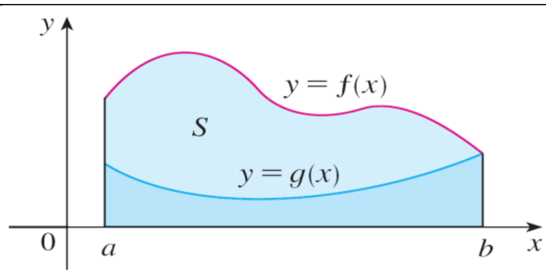


(a) Typical rectangle

(b) Approximating rectangles

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in the interval $[a, b]$ is

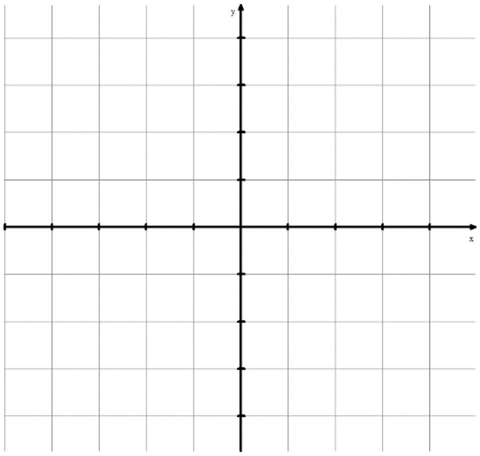
$$A = \int_a^b [f(x) - g(x)]dx$$



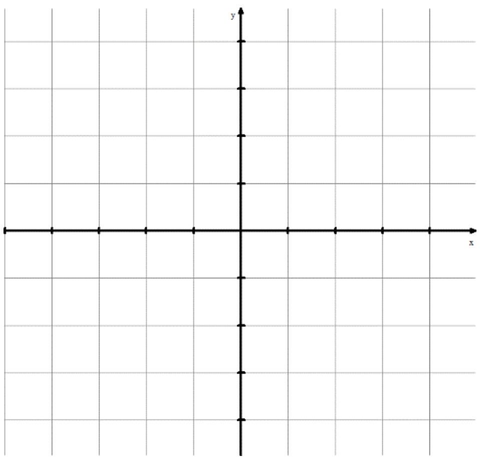
If f and g are both positive, then we can think about the area as

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$

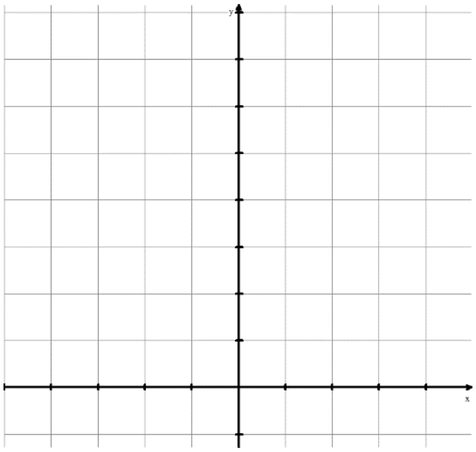
Example 1: Sketch the area between the curves $f(x) = \frac{1}{6}x^2 + 3$ and $g(x) = x$ on the interval $[-6, 6]$. Calculate this area using an integral.



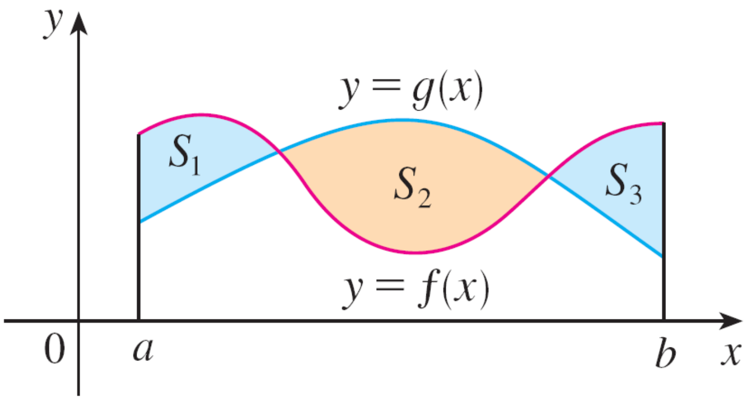
Example 2: Sketch the region bounded by the curves $f(x) = 6 - x^2$ and $y + x = 0$. Calculate this area using an integral**.**



Example 3: Sketch the region bounded by the curves $f(x) = 5x^2$ and $g(x) = 2x^2 + 75$. Calculate this area using an integral**.**



What happens if we are asked to find the area between two curves, but $f(x)$ is not always larger than $g(x)$?



If that is the case, we split up the area into regions where either $f(x) \geq g(x)$ or $g(x) \geq f(x)$ for the entire interval. Then find the area of each region and add them all up.

The area A between two curves $y = f(x)$ and $y = g(x)$, and the lines $x = a$ and $x = b$ is

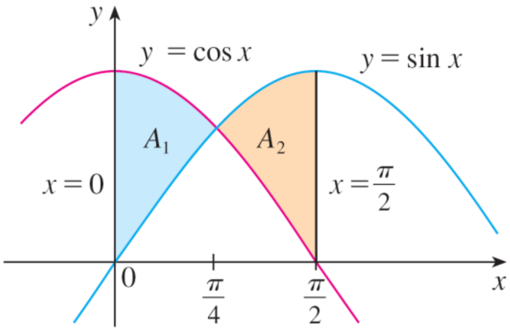
$$A = \int_a^b |f(x) - g(x)|dx$$

This means that we will most likely have to split up the integral every time $|f(x) - g(x)| = 0$

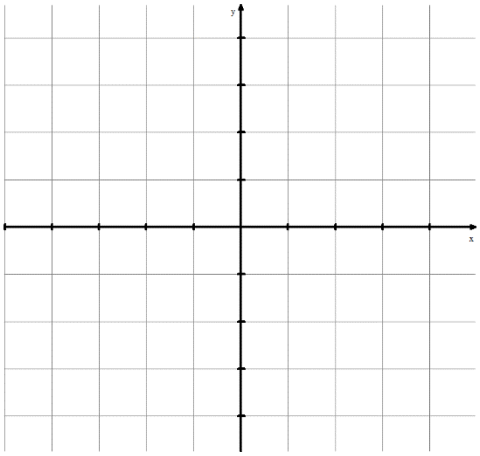
For example, let's say $f(x) \geq g(x)$ for $a \leq x \leq c$ and $g(x) \geq f(x)$ for $c \leq x \leq b$. Then

$$\int_a^b |f(x) - g(x)|dx = \int_a^c \big(f(x) - g(x)\big)dx + \int_c^b \big(g(x) - f(x)\big)dx$$

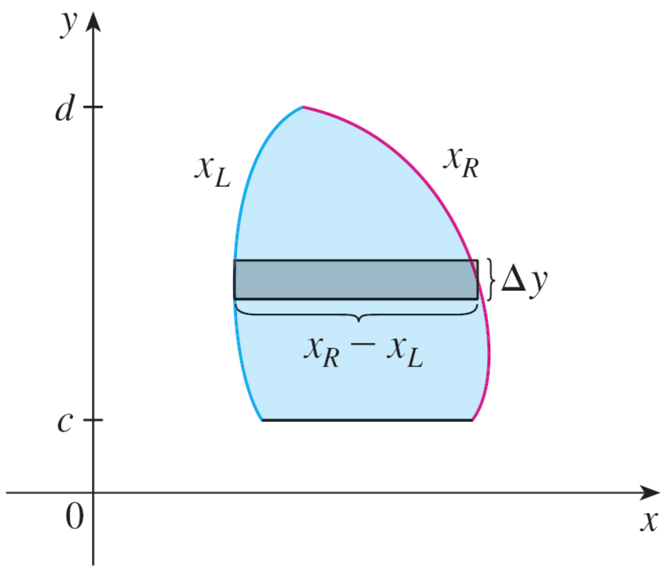
Example 4: Find the area bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$



Example 5: Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 - 2x = 6$.



Well, that sucked. Is there a better way? Why yes, of course there is! In the last example the bottom boundary of our region changed half through the problem. But if we instead look at the left and right boundaries, they don't change. So instead of integrating with respect to *x*, we should integrate with respect to *y*.



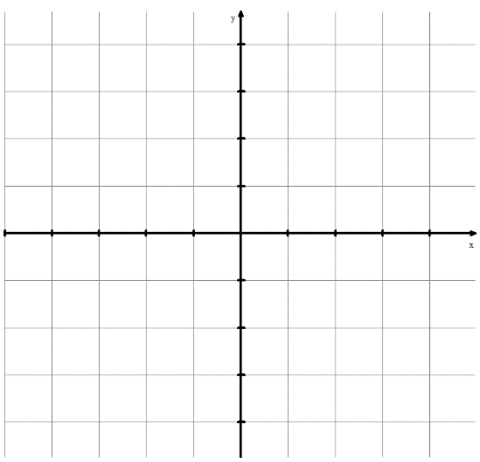
$$A = \int_c^d (X_R - X_L)dy$$

If we know the equations of the left and right boundaries as functions of *y*, then

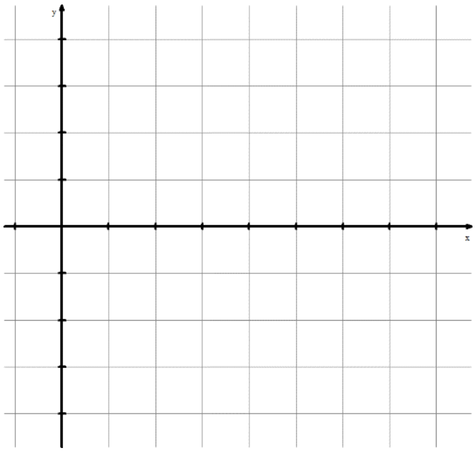
$$A = \int_c^d [f(y) - g(y)]dy$$

Steve's pro tip: It's usually best to try this method whenever you have an **even power of *y***

Example 6: Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 - 2x = 6$, but this time integrate using *y* as your independent variable.



Example 7: Sketch the region between the curves $f(y) = 2y^2$ and $g(y) = y^2 + 4$. Calculate this area using an integral.



Example 8: Sketch the region between the curves $x + y^2 = 12$ and $y + x = 0$ and decide if you should calculate the area between them using an integral with respect to *x* or *y*