Roulette:

On an American roulette wheel here are 38 compartments where the ball can land. They are numbered 1-36, and there are two compartments labeled 0 and 00. Half of the compartments numbered 1-36 are red and half are black. The compartments labeled 0 and 00 are green. On a French roulette table there is a single 0 but not the double 0, so there are 37 compartments where the ball can land. Here is a picture of a French roulette wheel.

For our discussion, we assume we are using an American roulette wheel. The "house odds" for betting on a number are 35: 1. If you bet one dollar on a number and win then you win $35, and if you lose then you have lost your dollar.

We first consider the game from the player's point of view: Suppose you have picked a number and placed your bet. The probability of winning is

$$p(E) = n(E)/n(S) = 1/38.$$

Now assume you play 38 times and win exactly once and lose the other 37 times. Your winnings are $35. and your losses are $37.

Your total winnings are then -2 dollars. In other words, you have lost $2. Since you played 38 times, your average "winnings" are $-2/38$ which is approximately $-\$0.053$, or a little more than 5 cents lost per bet.

If you consider the game from the house point of view, then the house makes about 5.3 cents per bet for each dollar bet.

Expected Value: From the player's point of view, the probability of winning, is $p(E) = 1/38$ and the probability of losing is $p(E') = 37/38$. The value of winning is 35 dollars and the value of losing is one dollar.

The expected value is the value of winning times the probability of winning plus the value of losing times the probability of losing.

$$E.V. = 35 \times p(E) + (-1) \times p(E') = 35 \times (1/38) - 1 \times (37/38) = -2/38 \approx \$ - 0.053$$

as before.

In general, suppose there are outcomes $E_1, \ldots E_n$ with probabilities $p(E_1), \ldots p(E_n)$ and with values $v(E_1) \ldots v(E_n)$. The expected value is the sum of the values of the outcomes times the probability of the outcomes:

$$E.V = v(E_1) \times p(E_1) + \cdots + v(E_n) \times p(E_n)$$

Example

Based on sale records, a salesperson knows the weekly commissions have the following probabilities:

Commision (in dollars)	0	1000	2000	2500	3000
Probability	0.1	0.25	0.35	0.2	0.1

What is the expected commission each week?

Answer:

$$E.V. = 0 \times (0.1) + 1000 \times (0.25) + 2000 \times (0.35) + 2500 \times (0.2) + 3000 \times (0.1) = \$1750$$

Another Example:

There are three one-dollar bills and two ten-dollar bills in a box. You are allowed to draw one bill at random and keep it. What is the expected value of the game? This amount could be considered the price you should pay for playing the game.

Answer:

The probability of drawing a one-dollar bill is 3/5 and the probability of drawing a ten-dollar bill is 2/5. The expected value is

$$E.V. = 1 \times (3/5) + 10 \times (2/5) = \$4.60$$

Terminology: If you decide to play a game because your expected value is positive, then that is referred to as using decision theory. If you use decision theory then you will never gamble in a casino.

Another Example:

There are seven one-dollar bills and two ten-dollar bills in a box. You are allowed to draw two bills at random and keep them. In order to play the game, you must buy a ticket for \$8. Based on decision theory, should you play the game?

Answer:

There are three outcomes possible:
Call E_1 the outcome that you get two one-dollar bills.
Call E_2 the outcome that you get one one-dollar bill and one ten-dollar bill.
Call E_3 the outcome that you get two ten-dollar bills.

The probability of getting two one-dollar bills is the number of ways of selecting 2 from the 7 one-dollar bills divided by the total number of ways of choosing 2 bills from the 9 bills:
$p(E_1) = \dfrac{{}_7C_2}{{}_9C_2}$ and the value $v(E_1) = 2 - 8 = -6$.

There are 7×2 ways to choose one one-dollar bill and one ten-dollar bill, so
$p(E_2) = \dfrac{7 \times 2}{{}_9C_2}$ and the value $v(E_2) = 11 - 8 = 3$.

There is only one way to choose 2 ten-dollar bills, so
$p(E_3) = \dfrac{1}{{}_9C_2}$ and the value $v(E_3) = 20 - 8 = 12$.

$$E.V = v(E_1) \times p(E_1) + v(E_2) \times p(E_2) + v(E_3) \times p(E_3) = \frac{{}_7C_2}{{}_9C_2} \times (-6) + \frac{7 \times 2}{{}_9C_2} \times 3 + \frac{1}{{}_9C_2} \times 12$$

Carry out the arithmetic and find that

$$E.V. = -2$$

so you expect your average loss to be \$2 each time you play the game. The answer is no.

EXPECTED NUMBER OF TRIALS UNTIL SUCCESS:

Suppose you toss a die until you get a 3. Suppose you keep track of how many tosses it took until you got the 3. Sometimes you might get a 3 on the first toss. Other times it might take a lot of tosses. If you performed the experiment hundreds of times, what would you expect the average number of tosses (until you get a 3) to be?

Solution:
You would probably guess that you would get a 3 one out of every six times on the average, and that on the average you would get a 3 on the sixth toss. Of course, paradoxically you could make the same argument and conclude that the expected number of trials until you get a 4, or any of $\{1, 2, 3, 4, 5, 6\}$ is also 6. The paradox lies in the unusual use of the word "expected value."

We use our formula to compute the expected number of trials until a success for this example:

Let E_1 be the event that the you get a three on the first toss.
Let E_2 be the event that the first toss is not a three and the toss gives a 3.
Let E_3 be the event that the first two tosses don't give 3 but the third toss does.
Let E_4 be the event that the first fourth toss gives a 3 (but you don't get a 3 on any of the earlier tosses.)
Let E_5 be the event that the the fifth toss gives a 3 (but you don't get a 3 on any of the earlier tosses.)
Continue in this fashion.

$p(E_1) = \frac{1}{6}$ and $v(E_1) = 1$

$p(E_2) = \frac{5}{6} \times \frac{1}{6}$ and $v(E_2) = 2$

$p(E_3) = (\frac{5}{6})^2 \times \frac{1}{6}$ and $v(E_3) = 3$

$p(E_4) = (\frac{5}{6})^3 \times \frac{1}{6}$ and $v(E_1) = 4$

$p(E_5) = (\frac{5}{6})^4 \times 16$ and $v(E_5) = 1$

etc.
Substitute these values into

$$E.V. = p(E_1) \times v(E_1) + p(E_2) \times v(E_2) + p(E_3) \times v(E_3) + p(E_4) \times v(E_4) + p(E_5) \times v(E_5) + \dots$$

If you take more and more terms, your answer will get closer and closer to 6.

In general, suppose the probability of an event is p, hence the probability of a failure is $1 - p$. Suppose you want to know the expected number of trials until you have a success.
Let E_1 be the event that you get a success on the first trial.

Let E_2 be the event that you get a success on the second trial .
Let E_3 be the event that you get a success on the third trial.
Continue in this fashion.

$p(E_1) = p$ and $v(E_1) = 1$

$p(E_2) = (1-p) \times p$ and $v(E_2) = 2$

$p(E_3) = (1-p)^2 \times p$ and $v(E_3) = 3$

$p(E_4) = (1-p)^3 \times p$ and $v(E_1) = 4$

etc.

Substitute these values into

$$E.V. = p(E_1) \times v(E_1) + p(E_2) \times v(E_2) + p(E_3) \times v(E_3) + p(E_4) \times v(E_4) + p(E_5) \times v(E_5) + \dots$$
$$= p \times 1 + (1-p) \times p \times 2 + (1-p)^2 \times p \times 3 + (1-p)^3 \times p \times 4 + \dots$$

An argument using geometric series* can be used to show that

$$E.V. = \frac{1}{p}$$

Therefore

If p is the probability of a success then the expected number of trials until a success is $\frac{1}{p}$

END OF LECTURE

*The geometric series argument:

$$\begin{aligned} E.V. &= p(E_1) \times v(E_1) + p(E_2) \times v(E_2) + p(E_3) \times v(E_3) + p(E_4) \times v(E_4) + p(E_5) \times v(E_5) + \dots \\ &= p \times 1 + (1-p) \times p \times 2 + (1-p)^2 \times p \times 3 + (1-p)^3 \times p \times 4 + \dots \\ &= p(1 + (1-p) \times 2 + (1-p)^2 \times 3 + (1-p)^3 \times 4 + (1-p)^4 \times 5 + \dots) \end{aligned}$$

Multiply both sides by $(1-p)$

$E.V. \times (1-p) = p \times ((1-p) + (1-p)^2 \times 2 + (1-p)^3 \times 3 + (1-p)^4 \times 4 + \dots)$

Now subtract: $E.V. - (1-p) \times E.V. = p(1 + (1-p) + (1-p)^2 + (1-p)^3 + \dots)$

Now notice that $(1 + (1-p) + (1-p)^2 + (1-p)^3 + \dots) = \dfrac{1}{1-(1-p)}$ since it is a geometric series with $|1-p| < 1$.

So we have

$E.V. \times p = p \times \frac{1}{1-(1-p)} = 1$

Hence

$$E.V. = \frac{1}{p}$$