Math 869 Take–Home Final Exam

The rules: You may use your class notes, our textbook, or any other written resource. However, you may not discuss these problems with anyone (except me) before you hand them in.

Due date: Wednesday, May 2, at 5:00 PM in my office (A307).

1. Let X be a topological space, and let $f : I \to X$ and $g : I \to X$ be loops based at $x_0 \in X$. By gluing together the endpoints of an interval, f and and g give rise to continuous maps $\overline{f} : S^1 \to X$ and $\overline{g} : S^1 \to X$. Prove that $\overline{f}$ and $\overline{g}$ are homotopic (without regard to basepoints) if and only if f and g represent conjugate elements of $\pi_1(X, x_0)$.

2. Let $\varphi : (X, x_0) \to (X, x_0)$ be a basepoint–preserving homeomorphism. Define the *mapping torus* of φ to be the topological space

$$V_\varphi = (X \times I)/\sim, \quad \text{where} \quad (x, 0) \sim (\varphi(x), 1).$$

For example, if $X = S^1$ and φ is orientation–reversing, then V_φ will be the Klein bottle.

Give a presentation for $\pi_1(V_\varphi)$ in terms of $G = \pi_1(X, x_0)$. Your answer should depend only on G and the induced isomorphism $\varphi_* : G \to G$. (If it helps, you may assume that X is a CW complex and φ preserves the CW structure.)

3. Let $p : X \to Y$ be a covering map. Proposition 1.31 says that the induced homomorphism of fundamental groups is injective. Is the same true for $p_* : H_1(X) \to H_1(Y)$? What about the other homology groups?

4. Let M_g be an orientable surface of genus g. Compute the homology groups of $M_g \times S^2$.

5. Do exercise 38 on page 159. Note that by the observation at the top of page 162, this gives a way to derive the Mayer–Vietoris sequence from the axioms for homology.