Clarification and Extra Problems for Homework 10

Math 461, Fall 2006

8.4.3. It is worth emphasizing that the spaces X and Y in this problem may not be metric spaces. (Thus Proposition 16 and the corresponding theorem from class do not apply.) For the sequential compactness part of the problem, you may assume that X and Y are Hausdorff. Recall that we defined convergence of sequences in Hausdorff spaces as follows: the sequence (x_n) converges to x if and only if every neighborhood of x contains all but finitely many points of the sequence.

Extra Problems

1. Let (X, D) be a metric space.

(a) Suppose that for some $\epsilon > 0$, every neighborhood $N(x, \epsilon)$ has compact closure. Prove that X is complete. (Note that this implies $\mathbb{R}^n$ is complete.)

(b) Suppose that for each $x \in X$, there is an $\epsilon(x) > 0$ such that $N(x, \epsilon(x))$ has compact closure. Show by means of an example that X need not be complete.

2. Let (X, d) be a compact metric space, and let $\mathcal{C}(X)$ be the set of all continuous functions from X to $\mathbb{R}$. Define a metric D on $\mathcal{C}(X)$ by

$$D(f, g) = \max\{|f(x) - g(x)| : x \in X\}.$$

(You do not need to check that this is a metric.) Using the following outline, prove that $(\mathcal{C}(X), D)$ is complete.

(a) Let (f_n) be a Cauchy sequence in $\mathcal{C}(X)$. Prove that for any $x \in X$, $(f_n(x))$ is a Cauchy sequence in X.

(b) Define a function $f(x) = \lim f_n(x)$. Explain why this is well-defined.

(c) Prove that f_n converges uniformly to f.

(d) Prove that $f(x)$ is continuous – that is, $f \in \mathcal{C}(X)$. Now, observe that uniform convergence means that f_n converges to f in the metric D.