Review Questions for Midterm 2

Math 320, Fall 2006

1. You should know the definitions of the following terms. 4 of them will appear on the test.

- limit point
- isolated point
- open set
- closed set
- closure
- bounded set
- compact set
- connected set[1]
- $\lim_{x\to c} f(x)$
- $\lim_{x\to c^-} f(x)$, $\lim_{x\to c^+} f(x)$
- $f(x)$ is continuous at c
- $f(x)$ is uniformly continuous

2. Are the following true or false? Give a brief explanation or a counterexample.

- If $A \subseteq \mathbb{R}$ is not open, then A is closed.
- An open set cannot contain any isolated points.
- If A is a bounded set, then $\sup A$ is a limit point of A.
- Every non-empty compact set contains a non-empty open set.
- If $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c} \sqrt{f(x)^2+1}$ exists also.
- A decreasing function must be 1–1.
- If A is a closed set and $f(x)$ is continuous on A, then $f(A)$ is closed also.
- If A is a closed set and $f(x)$ is continuous and increasing on A, then $f(A)$ is closed also.
- If A is an open set and $f(x)$ is continuous and increasing on A, then $f(A)$ is open also.
- If A is a bounded interval and $f(x)$ is uniformly continuous on A, then $f(A)$ is a bounded interval.
- If $f(x)$ and $g(x)$ are uniformly continuous on A, $f(x)g(x)$ is uniformly continuous on A.
- There is a function $f : \mathbb{R} \to \mathbb{R}$ whose set of discontinuity is exactly $\mathbb{I} \cup \mathbb{Z}$.

[1] The book's definition of *connected* is a bit convoluted. You can use the following equivalent definition: E is connected if whenever $a < c < b$ and $a, b \in E$, then $c \in E$ also.

3. Prove that the Cantor set C is compact.

4. Prove that the intersection of finitely many open sets is open. Is the intersection of infinitely many open sets necessarily open?

5. Prove that the function $f(x) = \dfrac{|x|}{x^2+1}$ is continuous on $\mathbb{R}$. *Hint: what theorem from the book makes this task much easier?*

6. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that

$$S = \{x \in \mathbb{R} \,:\, g(x) \in [0,1]\}$$

is a closed set.

7. Construct an *increasing* function $h : \mathbb{R} \to \mathbb{R}$, whose set of discontinuity is $\{\frac{1}{n} : n \in \mathbb{N}\}$.