Review Questions for Midterm 1

Math 320, Fall 2006

1. You should know the definitions of the following terms. 4 of them will appear on the test.

- upper bound, lower bound
- inf, sup
- 1–1, onto functions
- countable, uncountable
- convergence, divergence (for sequences)
- bounded sequence
- monotonic sequence
- Cauchy sequence
- subsequence
- convergence, divergence (for series)
- absolutely convergent series

2. Are the following true or false?

- If $\sup A < \sup B$, then some element of B is an upper bound for A.
- If $\sup A \leq \inf B$, and A does not have a maximum, then $a < b$ for all $a \in A$ and $b \in B$.
- Every subset of a countable set is countable.
- Every subset of an uncountable set is uncountable.
- If the sequences (a_n) and (b_n) converge, then $(a_n b_n)$ converges.
- If the sequences (a_n) and (b_n) diverge, then $(a_n b_n)$ diverges.
- Every bounded, monotonic sequence is Cauchy.
- There is a sequence (a_n), such that every $q \in \mathbb{Q}$ is the limit of a subsequence of (a_n).
- If $(a_n) \to \ell$, and (b_n) is a rearrangement of (a_n), then $(b_n) \to \ell$.
- If $\sum a_n$ converges, and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges.

3. Cardinality questions:

- Prove that the set of odd natural numbers is countable.
- Prove that the union of two disjoint, countable sets is countable.
- Prove that the intervals $(0, 1)$ and $(1, \infty)$ have the same cardinality as $\mathbb{R}$. *Hint: can you construct 1–1, onto functions among these three sets?*

4. Prove that $\inf\{2 + \frac{1}{n} : n \in \mathbb{N}\} = 2$. What theorems do you need for this proof?

5. Prove, in two essentially different ways, that $\mathbb{Q} \neq \mathbb{R}$. How many different ways can you prove this?

6. Consider the sequence (a_n), defined by $a_n = \dfrac{2n-1}{5n+2}$.

- Use your intuition to decide what $\lim(a_n)$ should be.
- Use the *definition* of convergence to prove that the limit exists, and is equal to your answer.
- Use the algebraic limit theorem, and the fact that $\frac{1}{n} \to 0$, to prove that the limit exists and to compute it.

7. Prove that every convergent sequence is bounded. *This is a theorem in the book, but you should know how to prove this without quoting the theorem.*

8. Prove that every convergent sequence has a monotonic subsequence.

9. Prove that $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n$ converges. There are at least three ways to do this!