Midterm Exam 2

Math 320-02, Fall 2006

You have 50 minutes. No notes, no books, no calculators. Good luck!

Name: Solutions ______________________

ID #: ______________________

1. ________ (/20 points)

2. ________ (/30 points)

3. ________ (/25 points)

4. ________ (/25 points)

Total ________ (/100 points)

Homework Average ________

Course Average ________

1. [20 points] State the definitions of the following terms or expressions.

(a) limit point of a set

Let $A \in \mathbb{R}$, ~~a~~ $x \in \mathbb{R}$. Then x is a limit point of A if for all $\varepsilon > 0$, the neighborhood $V_\varepsilon(x)$ intersects A in a point other than x.

(b) compact set

A set K is compact if every sequence contained in K has a convergent subsequence that converges to a limit in K.

(c) $\lim_{x \to c^+} f(x) = L$

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $0 < x - c < \delta$, then $|f(x) - L| < \varepsilon$.

(d) $f(x)$ is uniformly continuous

$f(x)$ is uniformly continuous on A if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for all $x, y \in A$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

2. [30 points] **True/False/Explain.** State whether each of the following statements is true or false. Then explain your answer, in one or two sentences. Provide a counterexample where it's relevant. *This problem does not need complete proofs – don't spend time writing them!*

(a) Every finite set is closed.

True. A finite set has no limit points, and thus contains all its (nonexistent) limit points.

(b) If K is a non-empty compact set, then $\sup K$ exists, and is contained in K.

True. If K is compact, it is closed and bounded. Since K is bounded, $\sup K$ exists. If $\sup K$ is a limit point, it must be in K because K is closed. Otherwise, it's an isolated point of K, and is thus also in K.

(c) Every non-empty open set is uncountable.

True. A non-empty open set contains an interval $V_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$. This interval has the same cardinality as $\mathbb{R}$, so the open set is uncountable.

True/False/Explain, continued.

(d) If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} f(x)g(x)$ both exist, then $\lim_{x \to c} g(x)$ exists also.

<u>False</u>. Let $c = 0$, $f(x) = x$, $g(x) = \frac{1}{x}$.

Then $\lim_{x \to c} f(x)g(x) = \lim_{x \to 0} (1)$ exists, and

$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x)$ also exists, but

$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{1}{x}$ does not exist.

(e) There is a non-empty compact set K and a continuous function $h : K \to \mathbb{R}$, such that $h(K)$ is an open set.

<u>False</u>. If K is compact and ~~h(x)~~ h is continuous, $h(K)$ is also compact. Thus $h(K)$ is closed, non-empty, and bounded (hence $h(K) \neq \mathbb{R}$). The only sets that are both open and closed are $\emptyset$ and $\mathbb{R}$.

(f) There is an increasing function $f : \mathbb{R} \to \mathbb{R}$ whose set of discontinuity D_f is the Cantor set.

<u>False</u>. D_f for an increasing function is finite or countable. On the other hand, C is uncountable.

3. [25 points] Let A and B be closed sets. Prove that $A \cup B$ is closed.

We need to prove that $A \cup B$ contains its limit points. So, let x be a limit point of $A \cup B$. Then there is a sequence $(x_n) \to x$, where $x_n \in A \cup B$ for all n. (x_n) must have a subsequence (x_{n_k}) that is contained in just A or just B. Then

$$\lim (x_{n_k}) = \lim (x_n) = x,$$

so x is a limit point of A or B. Since A and B are closed, $x \in A$ or $x \in B$. Therefore, $x \in A \cup B$, and $A \cup B$ is closed.

Alternate approach: use the definition of "limit point" instead of sequences.

Alternate approach: take complements of A and B, which are open sets. Then prove that the intersection of two open sets is open.

4. [25 points]

(a) Prove that the function $f(x) = 2|x|$ is uniformly continuous on $[-1, 1]$.

Choose an arbitrary $\varepsilon > 0$, and let $\delta = \varepsilon/2$.
Now, suppose that $x, y \in [-1, 1]$ and $|x - y| < \delta$.

$$\text{Then } |f(x) - f(y)| = |2|x| - 2|y|| \leq 2|x - y| < 2\delta = \varepsilon.$$

Thus $f(x)$ is uniformly continuous (and, in particular, continuous).

(b) Use part (a) to prove that $g(x) = \dfrac{2|x|}{4|x| + 1}$ is uniformly continuous on $[-1, 1]$.

Since $f(x)$ is continuous on $[-1, 1]$,
$4|x| = 2f(x)$ is continuous

$\Rightarrow$ $4|x| + 1$ is continuous (and $4|x| + 1 \neq 0$ on $[-1, 1]$)

$\Rightarrow$ $\dfrac{2|x|}{4|x|+1} = \dfrac{f(x)}{2f(x)+1}$ is continuous on $[-1, 1]$.

Since $[-1, 1]$ is a compact set and $g(x)$ is continuous on $[-1, 1]$, it must be uniformly continuous.

(Note: combinations of uniformly continuous functions are not automatically uniformly continuous.)