Midterm Exam 1

Math 320-02, Fall 2006

You have 50 minutes. No notes, no books, no calculators. Good luck!

Name: Solutions ____________________

ID #: ________________________________

1. __________ (/20 points)

2. __________ (/30 points)

3. __________ (/25 points)

4. __________ (/25 points)

Total __________ (/100 points)

Homework Average __________

Course Average __________

1. [20 points] State the definitions of the following terms or expressions.

(a) $\inf A$

$\ell = \inf A$ if

- ℓ is a lower bound on A (for all $a \in A$, $\ell \le a$)
- every lower bound b for A satisfies $b \le \ell$.

(b) A set S is *uncountable.* (Please give a direct definition of this, without relying on the notion of *countable.*)

S is uncountable if S is infinite and there does not exist a 1-1, onto function $f: \mathbb{N} \to S$.

(c) *Cauchy sequence*

A sequence (a_n) is Cauchy if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. for all $m, n \ge N$, $|a_m - a_n| < \varepsilon$.

(d) The series $\sum_{k=1}^{\infty} a_k$ *converges.*

Let $s_n = \sum_{k=1}^{n} a_k$, for each $n \in \mathbb{N}$. Then $\sum_{k=1}^{\infty} a_k$ converges if $(s_n) \to \ell$ for some $\ell \in \mathbb{R}$.

That is, ~~then~~ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n \ge N$, $|a_n - \ell| < \varepsilon$.

2. [30 points] **True/False/Explain.** State whether each of the following statements is true or false. Then explain your answer, in one or two sentences. Provide a counterexample where it's relevant. *This problem does not need complete proofs – don't spend time writing them!*

(a) If $\sup A \leq \sup B$, then some number $b \in B$ is an upper bound for A.

False. For example, $A = B = (0,1)$. Then $\sup A = \sup B = 1$, but no number $b \in B$ is an upper bound for A, since $A = B$ does not have a maximum.

(b) The union of countably many countable sets is countable.

True. This was Theorem 1.4.13.

(c) If the sequence (a_n) diverges, then $(1/a_n)$ converges.

False. For example, $a_n = (-1)^n$. Then $\frac{1}{a_n} = a_n = (-1)^n$, so $\left(\frac{1}{a_n}\right)$ also diverges.

True/False/Explain, continued.

(d) There exists a sequence (b_n), with the property that every real number $x \in [0, 1]$ is the limit of a subsequence of (b_n).

True. For example, we can use the sequence
$(b_n) = (0.1, 0.2, \ldots, 0.9, 0.01, 0.02, \ldots, 0.99, 0.001, \ldots, 0.999, \ldots)$
Then every decimal expansion (hence every number $x \in [0,1]$) is the limit of a subsequence of (b_n).

Alternately, we could use the fact that $\mathbb{Q}$ is countable to arrange the rational numbers in a sequence.

(e) Every bounded sequence contains a monotonic subsequence.

True. By Bolzano-Weierstrass, a bounded sequence (a_n) has a convergent subsequence (a_{n_k}). If $\ell = \lim(a_{n_k})$, then the terms of (a_{n_k}) either to the left or right of ℓ form a monotonic subsequence of (a_n).

$$\cdots\, a_{n_k} \quad \overset{\ell}{\downarrow} \quad \underbrace{\cdots\, a_{n_k}}_{} \text{ monotonic subsequence.}$$

(f) If the series $\sum a_n$ converges, then $\sum a_n^2$ also converges.

False. For example, if $a_n = \frac{(-1)^n}{\sqrt{n}}$, then $\sum a_n$ converges by the Alternating Series Test. On the other hand, $\sum a_n^2 = \sum \frac{1}{n}$ diverges.

3. [25 points] Let $S = \left\{ \frac{n-1}{2n} : n \in \mathbb{N} \right\}$.

(a) Find $\sup S$.

$\sup S = \frac{1}{2}$.

(b) Prove that your answer is, in fact, the supremum of S.

There are two properties to check:

1) Because $\frac{n-1}{2n} < \frac{n}{2n} = \frac{1}{2}$ for all n, $\frac{1}{2}$ is an upper bound for S.

2) Let $b \in \mathbb{R}$ be an upper bound for S. Suppose, for a contradiction, that $b < \frac{1}{2}$. Then $\frac{1}{2} - b > 0$. By the Archimedean Property, there is an $n \in \mathbb{N}$ s.t.

$$0 < \frac{1}{n} < \frac{1}{2} - b.$$

In particular, $0 < \frac{1}{2n} < \frac{1}{2} - b$.

Then $b < \frac{1}{2} - \frac{1}{2n}$

$$b < \frac{n}{2n} - \frac{1}{2n}$$

$$b < \left(\frac{n-1}{2n}\right) \in S. \quad \text{(Contradiction.)}$$

Thus any number $b < \frac{1}{2}$ can't be an upper bound.

By 1 and 2, $\sup S = \frac{1}{2}$.

4. [25 points] Consider the sequence (a_n), where

$$a_n = (-1)^n \left(\frac{n-1}{2n}\right).$$

So $(a_n) = \left(\frac{0}{2}, \frac{1}{4}, \frac{-2}{6}, \frac{3}{8}, \frac{-4}{10}, \frac{5}{12}, \ldots\right)$

(a) Prove that (a_n) does not converge to any real number.

We will prove that (a_n) is not Cauchy.

Note that, for $n > 1$, $|a_n| = \frac{n-1}{2n} \geq \frac{1}{4}$.

Thus $a_n \geq \frac{1}{4}$ for n even,

$a_n \leq -\frac{1}{4}$ for n odd (and $n > 1$).

So, choose $\varepsilon = \frac{1}{2}$. Then, for any $N \in \mathbb{N}$, no matter how large N is, we can always find $m \geq N$ and $n \geq N$, m odd, n even, so that $|a_m - a_n| \geq \frac{1}{2} = \varepsilon$.

Since (a_n) is not Cauchy, it cannot converge.

(b) In a sentence or two, describe an alternate way to prove that (a_n) diverges. (*There are at least three different ways to do this problem.*)

Two alternate approaches:

① <u>From the definition</u>. For any putative limit $\ell \in \mathbb{R}$, we have to choose $\varepsilon > 0$ such that there are infinitely many terms with $|a_n - \ell| \geq \varepsilon$. $\varepsilon = 1/4$ will always work.

② <u>Subsequences</u>. We could prove that the subsequence of odd terms converges to $\frac{-1}{2}$, and the subsequence of even terms to $\frac{1}{2}$.