Midterm Exam 3

Math 132-06, Fall 2005

You have 50 minutes. No notes, no books, no calculators. **You must show all work to receive credit!** Good luck!

Name: Solutions

ID #: ______________________

1. ________ (/22 points)

2. ________ (/18 points)

3. ________ (/20 points)

4. ________ (/20 points)

5. ________ (/20 points)

Total ________ (/100 points)

Test Average ________

Gateway Points ________

Course Average ________

1. [22 points] A particle moves along the x-axis. At any time $t > 0$, its acceleration is

$$a(t) = \sin(t) + \frac{1}{4\sqrt{t}}.$$

(b) [8 points] When $t = 0$, the particle is standing still. What is its velocity at time t?

$$a(t) = v'(t) = \sin t + \frac{1}{4} t^{-1/2}$$

$$v(t) = -\cos(t) + \frac{1}{2} t^{1/2} + C$$

$$v(0) = 0 \Rightarrow 0 = -\cos(0) + \frac{1}{2}\cdot 0^{1/2} + C_1$$

$$0 = -1 + C_1$$

$$C_1 = 1$$

$$v(t) = -\cos(t) + \frac{1}{2} t^{1/2} + 1$$

(b) [8 points] When $t = 0$, the particle is at the origin. What is its position at time t?

$$p'(t) = v(t) = -\cos(t) + \tfrac{1}{2} t^{1/2} + 1$$

$$p(t) = -\sin(t) + \tfrac{1}{3} t^{3/2} + t + C_2$$

$$p(0) = 0 \Rightarrow 0 = -\sin(0) + \tfrac{1}{3}\cdot 0^{3/2} + 0 + C_2$$

$$C_2 = 0$$

$$p(t) = -\sin(t) + \frac{1}{3} t^{3/2} + t$$

(c) [6 points] In the time interval $[0, 4]$, what is the maximum distance from the origin that the particle will reach? (You can leave the answer in terms of trig functions, roots, etc.)

On $[0, 4]$, the velocity $v(t)$ is always non-negative
This is because $1 - \cos(t) \geq 0$ and $\frac{1}{2} t^{1/2} \geq 0$.
In fact, $v(t) = 0$ only when $t = 0$.
So the particle never turns around.
Its maximum distance will be at $t = 4$.

$$p(4) = -\sin(4) + \frac{1}{3}(4)^{3/2} + 4$$

$$p(4) = -\sin(4) + \frac{1}{3}(8) + 4$$

$$\boxed{p(4) = -\sin(4) + \frac{20}{3}}$$

2. [18 points] Compute the following limits, using any method discussed in class.

(a) [6 points] $\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{2x^2}$

$\lim_{x\to\infty} \sqrt{x^2+5} = \infty$ ✓

$\lim_{x\to\infty} 2x^2 = \infty$ ✓.

Can use L'Hopital.

$$\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{2x^2} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{2x}{2\sqrt{x^2+5}}}{4x} = \lim_{x\to\infty} \frac{\cancel{2x}}{2\sqrt{x^2+5}} \cdot \frac{1}{\cancel{4x}\ 2}$$

$$= \lim_{x\to\infty} \frac{1}{4\sqrt{x^2+5}} = 0.$$

(b) [6 points] $\lim_{x\to 1} \frac{x^a - 1}{x^b - 1}$

As $x \to 1$, $x^a - 1 \to 0$ ✓

$x^b - 1 \to 0$ ✓

$$\lim_{x\to 1} \frac{x^a - 1}{x^b - 1} \overset{L'H}{=} \lim_{x\to 1} \frac{a x^{a-1}}{b x^{b-1}} = \frac{a \cdot 1^{a-1}}{b \cdot 1^{b-1}} = \frac{a}{b}.$$

(c) [6 points] $\lim_{x\to 0} \frac{x \sin(x)}{1-\cos(x)}$

As $x \to 0$, $x \sin(x) \to 0$ ✓

$1 - \cos(x) \to 0$ ✓.

$$\lim_{x\to 0} \frac{x \sin(x)}{1-\cos(x)} \overset{L'H}{=} \lim_{x\to 0} \frac{\sin(x) + x\cos(x)}{\sin(x)} \quad \left(\text{"}\frac{0}{0}\text{" again}\right)$$

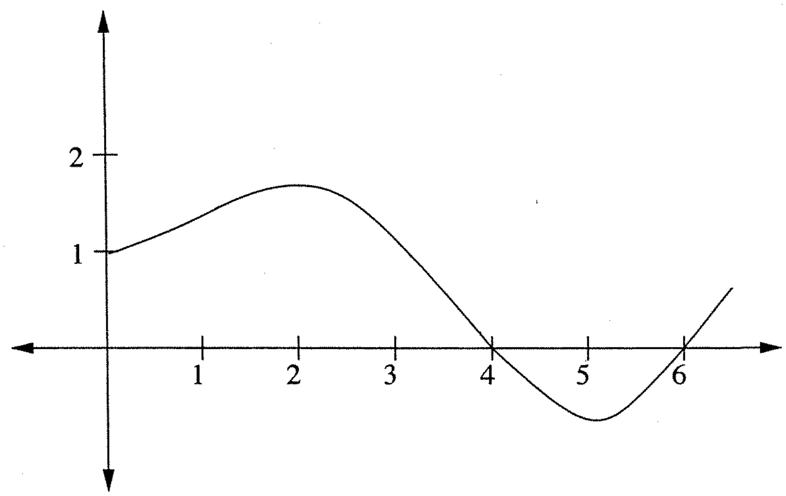
$$\overset{L'H}{=} \lim_{x\to 0} \frac{\cos(x) + \cos(x) + x\sin(x)}{\cos(x)}$$

$$= \frac{2\cos(0) + 0\cdot\sin(0)}{\cos(0)}$$

$$= \frac{2}{1}$$

$$= 2.$$

3. [20 points] The following is a graph of $f'(x)$. *Note that this is a graph of the derivative, not of* $f(x)$.



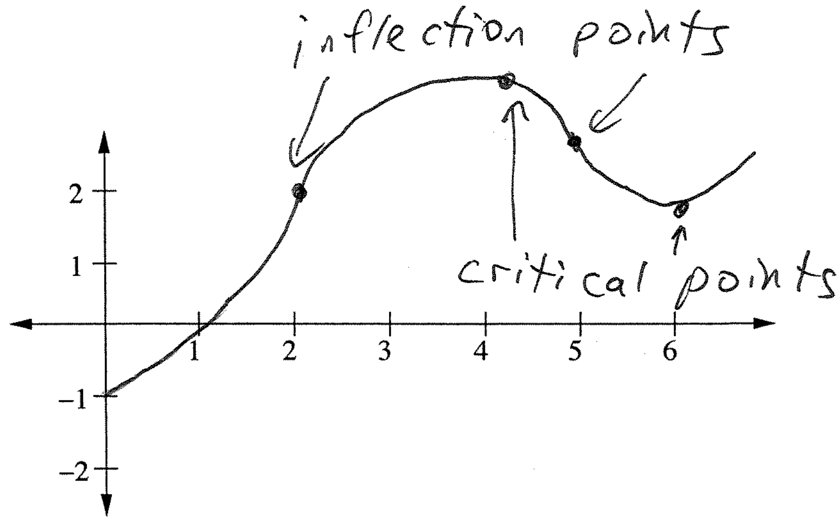
(a) [6 points] Where is $f(x)$ increasing? Where is it decreasing? Where are its critical points?

Increasing when $f'>0$: on $(0,4)$ and $(6,7)$
Decreasing when $f'<0$: on $(4,6)$
Critical points at $x=4$, $x=6$.

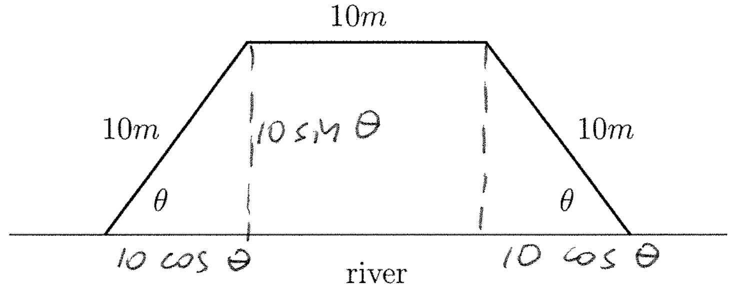
(a) [6 points] Where is $f(x)$ concave up? Where is it concave down? Where are its inflection points?

Concave up when f' is increasing: on $(0,2)$, $(5,7)$
Concave down when f' is decreasing: on $(2,5)$
Inflection points at $x=2$, $x=5$.

(c) [8 points] Suppose $f(0) = -1$. Sketch a graph of $f(x)$. Label all ciritical points and points of inflection.



4. [20 points] A farmer has a plot of land next to a straight riverbank. He has three sections of fence, each 10 meters long, and wants to build a pen next to the river using these pieces. The pen will come out in the shape of a trapezoid:



(a) [6 points] Given angle θ in the interval $[0, \frac{\pi}{2}]$, what is the area of the pen?

$$\text{Area} = A(\text{rectangle}) + 2 \cdot A(\text{triangle})$$
$$= 10 \cdot 10 \sin\theta + 2 \cdot \tfrac{1}{2} \cdot 10\cos\theta \cdot 10\sin\theta$$
$$= 100\sin\theta + 100\cos\theta\sin\theta$$
$$= (100 + 100\cos\theta) \cdot \sin\theta$$

(b) [8 points] What angle θ will maximize the area of the pen? What will that area be?

$$A'(\theta) = (100 + 100\cos\theta)\cdot\cos\theta + (-100\sin\theta)\sin\theta$$
$$= 100\cos\theta + 100\cos^2\theta - 100\sin^2\theta$$
$$= 100\cos\theta + 200\cos^2\theta - 100 \quad [-\sin^2\theta = \cos^2\theta - 1$$

$$2\cos^2\theta + \cos\theta - 1 = 0$$
$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$
$$\cos\theta = 1/2 \quad \text{or} \quad \cos\theta = -1$$
$$\theta = \pi/3 \quad \text{or} \quad \theta = \pi$$

not in the interval.

So $\theta = \pi/3$

$\cos\theta = 1/2 \quad \sin\theta = \frac{\sqrt{3}}{2}$.

$$\text{Area} = (100 + 100 \cdot \tfrac{1}{2}) \cdot \frac{\sqrt{3}}{2}$$
$$= 150 \cdot \frac{\sqrt{3}}{2} = \boxed{75\sqrt{3}}$$

(c) [6 points] Explain, using the theorems discussed in class, why you know this is the maximum area.

The extreme value theorem says that the maximum will occur either at an endpoint of the interval or at a critical point. At the endpoint $\theta = 0$, the area is 0. At the endpoint $\theta = \pi/2$, the area is 100. Both are less than $75\sqrt{3}$, so this is the absolute maximum.

5. [20 points]

(a) [8 points] State the Mean Value Theorem.

Let f be a function that is continuous on $[a, b]$ and differentiable on the open interval (a,b). Then, for some x-value c in the interval,

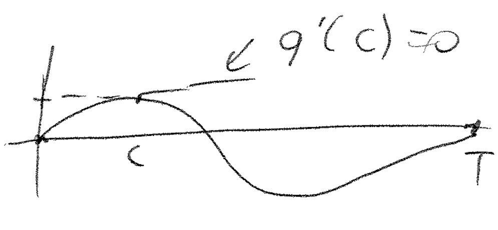
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

(Intuitively, the instantaneous rate of change, $f'(c)$, equals the average rate of change over the interval.)

(b) [12 points] Elena and Megan start a race at the same time and finish the race at the same time. Prove that, at some point during the race, they had the exact same speed.
Hint: consider the function $g(t)$ that describes the gap between Elena and Megan at time t.

$g(t)$ is continuous and differentiable (its derivative $g'(t)$ expresses the difference in Elena's and Megan's speeds).
So we can use the Mean Value Theorem.
The race starts at time 0 and ends at time T.
MVT says: for some c, $g'(c) = \frac{g(T)-g(0)}{T-0}$

$$= \frac{0-0}{T} = 0.$$



At this time c, the difference in speeds is 0, so Elena and Megan are going at the same speed.