Chapter 4 Review Problems

Math 132-06, Fall 2005

1. A particle moves back and forth along the x-axis. At time $t = 0$, it is at the origin and moving forward with velocity 3 units per second. At any time t, its acceleration is $3\cos(3t)$. Figure out its position at time t.

Given information: $s(0) = 0$

$v(0) = 3$

$a(t) = 3\cos(3t)$.

$a(t) = v'(t) = 3\cos(3t)$.

So $v(t) = \sin(3t) + C_1$

$v(0) = 3 \Rightarrow 3 = \sin(0) + C_1 \Rightarrow C_1 = 3$.

$v(t) = s'(t) = \sin(3t) + 3$

So $s(t) = \frac{-1}{3}\cos(3t) + 3t + C_2$.

$s(0) = 0 \Rightarrow 0 = \frac{-1}{3}\cos(0) + 3 \cdot 0 + C_2$

$0 = \frac{-1}{3} + C_2 \Rightarrow C_2 = 1/3$.

$$\boxed{s(t) = \frac{-1}{3}\cos(3t) + 3t + 1/3}$$

2. Compute the following limits. (You can use L'Hôpital's rule whenever it applies.)

(a) $\lim_{x \to \pi} \frac{\sin x}{x}$

$\lim_{x \to \pi} \sin(x) = \sin(\pi) = 0$

$\lim_{x \to \pi} (x) = \pi$.

So L'Hôpital doesn't apply, but $\lim_{x \to \pi} \frac{\sin x}{x} = \frac{0}{\pi} = 0$.

(b) $\lim_{x \to 0} (\csc x - \cot x) = \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \lim_{x \to 0} \frac{1 - \cos x}{\sin x}$.

As $x \to 0$, $1 - \cos x \to 0$ ✓

$\sin x \to 0$ ✓.

So L'Hôpital applies.

$\stackrel{L'H}{=} \lim_{x \to 0} \frac{\sin x}{\cos x}$

$= \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0$.

3. Let f be a function which is continuous on the interval $[0,3]$. The following chart gives values of f, f', and f'' on this interval.

	$x=0$	$0<x<1$	$x=1$	$1<x<2$	$x=2$	$2<x<3$
$f(x)$	1	positive	0	negative	-1	negative
$f'(x)$	undefined	negative	0	negative	undefined	positive
$f''(x)$	undefined	positive	0	negative	undefined	negative

(a) Where is f increasing? Where is f decreasing?

increasing when $f'>0$: on $(2,3)$

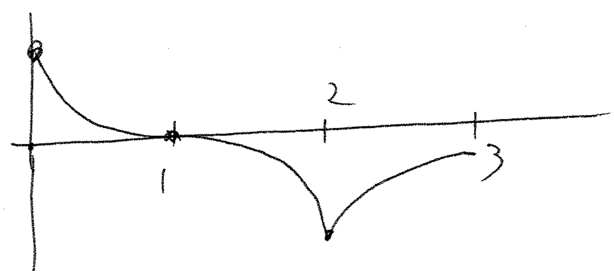
decreasing when $f'<0$: on $(0,1)$ and $(1,2)$

(b) Where is f concave up? Where is f concave down?

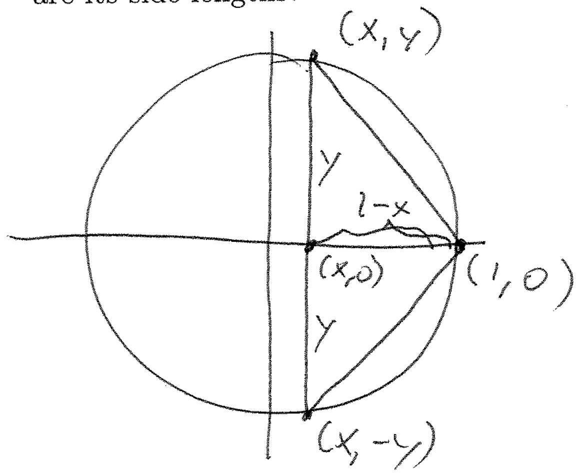
concave up when $f''>0$: on $(0,1)$

Concave down when $f''<0$: on $(1,2)$ and $(2,3)$

(c) Sketch a graph of what f might look like.



4. You want to construct an isosceles triangle whose vertices lie on the unit circle $x^2 + y^2 = 1$. One vertex of the triangle is at the point $(1, 0)$, while the other two vertices are at symmetric points above and below the x-axis. What triangle of this sort will have the largest area? What are its side lengths?



$x^2 + y^2 = 1$

$y^2 = 1 - x^2$

$y = \sqrt{1-x^2}$

$A = \frac{1}{2} \cdot b \cdot h$

$= \frac{1}{2} \cdot 2y \cdot (1-x)$

$= y \cdot (1-x)$

$= \sqrt{1-x^2}\,(1-x)$

$$A' = \sqrt{1-x^2}\,(-1) + \frac{-2x}{2\sqrt{1-x^2}}(1-x) = 0$$

$$\sqrt{1-x^2} = \frac{-x}{\sqrt{1-x^2}}(1-x)$$ ← Multiply both sides by $\sqrt{1-x^2}$

$1 - x^2 = -x(1-x)$

$1 - x^2 = -x + x^2$

$2x^2 - x - 1 = 0$

$(2x+1)(x-1) = 0$

$x = -\frac{1}{2}$ or $x = 1$.

$A = \sqrt{1 - \frac{1}{4}} \cdot \frac{1}{2}$

$= \sqrt{(3/4)} \cdot \frac{1}{2}$

$\boxed{A = \frac{\sqrt{3}}{4} \text{ maximum}}$

$A = 0$

(minimum)

Maximal area when $x = -\frac{1}{2}$, $y = \sqrt{3}/2$.

Sidelengths are:

base $= 2y = \sqrt{3}$.

diagonals: $d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

$= \sqrt{(\sqrt{3}/2 - 0)^2 + (-\frac{1}{2} - 1)^2}$

$= \sqrt{3/4 + (3/2)^2}$

$= \sqrt{3/4 + 9/4}$

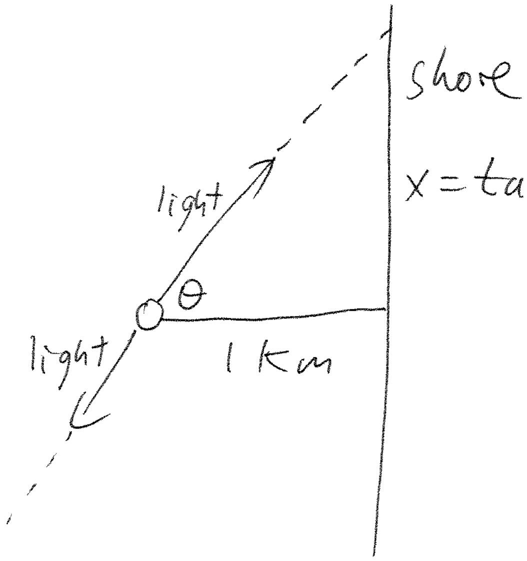
$= \sqrt{12/4} = \sqrt{3}$.

$\boxed{\text{Triangle is equilateral!!}}$

5. A straight shoreline runs north-south. A lighthouse, positioned 1 km offshore, shines a beam of light in two opposite directions. The beam of light rotates, moving along the shore from south to north, and makes a full turn every 2π minutes.

(a) Suppose that at time $t = 0$, the light shines directly at the closest point on the shore. Write down a formula for $f(t)$, the function that describes where on the shore the light beam hits at time t. What is $f(\frac{5\pi}{4})$?

(both θ and t make a full turn in 2π units, so they're the same).



$x = \tan\theta = \tan t$

$f(t) = \tan t.$

$f(0) = 0$ ✓.

$f(5\pi/4) = \tan(\frac{5\pi}{4}) = 1.$

(b) What is the average rate of change of $f(t)$ between $t = 0$ and $t = \frac{5\pi}{4}$? What is the instantaneous velocity of the illuminated spot at time t on this interval?

Average rate: $\dfrac{f(5\pi/4) - f(0)}{5\pi/4 - 0} = \dfrac{1-0}{5\pi/4} = \boxed{\dfrac{4}{5\pi}}$

Instantaneous rate: $f'(t) = \sec^2(t).$

(c) Will the instantaneous velocity ever equal the average velocity? Explain how this conclusion relates to the Mean Value Theorem.

NO. $f'(t) = \sec^2(t) = \dfrac{1}{\cos^2(t)} \geq 1$, because $\cos^2(t) \leq 1$.

On the other hand, $\frac{4}{5\pi} < 1$. So they can't be equal.

The mean value theorem doesn't apply here, because it only works for continuous functions (in fact, it needs f differentiable also). But $f(t) = \tan t$ is discontinuous at $x = \pi/2$, in the middle of the interval $[0, 5\pi/4]$. So the theorem doesn't apply to f.