RESEARCH STATEMENT

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My research lies at the interface of knot theory, three-dimensional topology, and hyperbolic geometry. More specifically, I seek to establish explicit connections between combinatorial descriptions of a manifold (such as a diagram of a knot or a triangulation of a 3–manifold) and invariants that come from geometry or algebra.

Thurston's geometrization conjecture, recently settled by Perelman, implies that every 3–manifold can be decomposed into relatively simple pieces, each of which admits one of eight homogeneous geometries. By far the richest of these is hyperbolic geometry, in which the manifolds have constant curvature -1. For the manifolds that do admit a hyperbolic metric, Mostow and Prasad showed that this metric is unique. Thus geometric measurements like volume and lengths of geodesics provide topological invariants. However, in the quarter-century since the formulation of Thurston's conjecture, finding effective and computable connections between the geometry and topology has proved to be a difficult problem, resistant to many attacks.

Most of the results described below fit into a theme of making the geometrization program more concrete and effective. I begin with combinatorial input. For example, a planar diagram describes a link and also its complement in 3–space; a triangulation or a Dehn surgery description specifies a 3–manifold. Given these data, I determine where this manifold fits into Thurston's framework: does it admit one of the eight geometries, or does it need to be cut into geometric pieces? In the generic case where the given manifold is hyperbolic, I use the combinatorial data to explicitly estimate its volume. In certain cases, it is possible to say more: Section 3.1 includes an infinite family of manifolds where one can explicitly construct the hyperbolic structure as a gluing of metric tetrahedra.

Some of my recent work has also uncovered new connections between quantum knot invariants and classical geometric topology. In particular, coefficients of the Jones polynomial control the behavior of certain spanning surfaces for a knot, and determine the volume of its complement up to a bounded factor. These results establish a coarse version of the Kashaev–Murakami volume conjecture; see Section 2.3 for more details.

1. Dehn surgery and geometry

Every closed 3–manifold can be constructed by a process called *Dehn surgery* or *Dehn filling.* Start with a manifold with boundary consisting of tori (such as the complement of a knot or link in S^3), and attach a solid torus to each boundary torus. The result depends only on the *slope* (isotopy class of simple closed curve) chosen to bound a disk in the solid torus. When the initial manifold is hyperbolic, there is a natural way to measure the *length* of this slope, using a Euclidean metric on the boundary torus. The length of the slope typically controls the outcome of the surgery. Thurston showed that if one starts with a hyperbolic manifold and performs Dehn filling along a sufficiently long slope, the resulting manifold will have a hyperbolic structure very close to the original [24]. Hodgson and Kerckhoff recently made this statement more quantitative [16].

1.1. Volume change under Dehn filling.

One of my recent results, proved jointly with Effie Kalfagianni and Jessica Purcell, is an explicit estimate on the change in volume under this operation [12]. Suppose M is a hyperbolic manifold and N is obtained by Dehn filling some or all of the boundary tori of M. If the shortest filling slope has length $\ell > 2\pi$, then we show that N will have a complete hyperbolic metric with

$$(1) \qquad \left(1-\left(\frac{2\pi}{\ell}\right)^2\right)^{3/2} \mathrm{vol}(M) \;\leq\; \mathrm{vol}(N) \;<\; \mathrm{vol}(M).$$

In the past year, equation (1) has found applications to both geometric and topological problems. Petronio and Vesnin have used it to bound the number of tetrahedra required to triangulate a particular family of manifolds [22]. More recently, Gabai, Meyerhoff, and Milley relied on our estimate in the endgame of their project to identify the smallest–volume closed hyperbolic 3–manifold [14]. In addition, equation (1) can be used to give diagrammatic estimates for the volumes of several large families of knots and links; see Section 2.2 below for details. We are confident that this result will find other applications in the future.

1.2. **Classifications of Dehn surgery space.** If one is given a particular manifold M with torus boundary, it is often possible to determine the geometric types of all the manifolds obtained by Dehn filling on M. For example, the surgeries on most small knots are completely understood. The task becomes harder when M has multiple boundary tori; one example of success in this task is Martelli and Petronio's classification of Dehn surgeries on a particular 3–component link [19]. A still harder problem is to classify all the Dehn fillings of an infinite family of manifolds; one rare example of a theorem of this type is Brittenham and Wu's classification of Dehn surgeries on two-bridge knots [2].

In recent joint work [9], Gabriel Indurskis, François Guéritaud, and I have classified the geometric types of all Dehn fillings of all hyperbolic *punctured torus bundles*. This type of bundle is constructed from $F \times I$, where F is a once-punctured torus, by gluing $F \times \{0\}$ to $F \times \{1\}$ via a prescribed homeomorphism. This homeomorphism of F, called the *monodromy map*, specifies a punctured torus bundle over the circle. Given the combinatorial data of a monodromy and a slope on the boundary torus, our theorem describes which of the eight homogeneous geometries the corresponding filled manifold will admit; in the rare cases where the filled manifold does not admit one of the eight geometries, we describe its geometric decomposition. This classification is a microcosm of geometrization: all eight geometries show up, but as usual the predominant geometry is hyperbolic.

The proof of our theorem is primarily knot-theoretic. It turns out that to understand the fillings of punctured torus bundles, it suffices to understand the double branched covers of links that arise as the closures of 3–string braids. If the double branched cover is not hyperbolic, all the obstructions to hyperbolicity (namely, essential spheres and tori, or a Seifert fibering) can be translated into conditions on the 3–string braid. Thus this theorem is a prime example of the interplay between knot theory, geometry, and topology. Studying the properties of knots helps us understand the geometry of closed manifolds; conversely, information about the closed manifolds tells us new information about the knots. Among the consequences of this theorem are a characterization of which 3–braids have hyperbolic complements and an estimate on their volumes.

2. Link diagrams and geometry

A major focus of my work has been the search for explicit connections between planar diagrams of a link and the geometry of the link complement. To quickly grasp the geometry of a link K, it helps to partition the crossings of a diagram into regions. A *twist region* is a section of a diagram $D(K)$ in which two strands of K wrap around each other maximally. Equivalently, a twist region consists of either a sequence of bigons or a single crossing that is not adjacent to any bigons. The number of these regions is called the *twist number* of $D(K)$, and is denoted $\mathrm{tw}(D)$. It turns out that in a *reduced* diagram, where obvious redundancies have been eliminated, this easy-to-read quantity carries a great deal of information about the geometry and topology of the link complement.

2.1. **Geometric type and surgery information.** Given a reduced diagram $D(K)$, it is often possible to tell whether the complement $S^3 \smallsetminus K$ is hyperbolic. For example, Menasco showed that an alternating link is hyperbolic if and only if its diagram is prime and has two or more twist regions [20]. Jessica Purcell and I extended this theorem to links with a lot of twisting: if a prime diagram has two or more twist regions, and at least six crossings in each region, then the given link is hyperbolic [13]. Using similar methods, one can also characterize the hyperbolic links within the families of arborescent links ([8], with Guéritaud) and closed 3–braids ([9], with Guéritaud and Indurskis).

Perhaps more surprisingly, a close study of the twisting in a diagram $D(K)$ also yields a wealth of information about the manifolds obtained by Dehn surgery on K. Purcell and I showed that when a reduced diagram of a knot K has at least four twist regions and at least six crossings in each region, every *non-trivial* surgery on K will yield a hyperbolic manifold [13]. (Trivial surgery involves removing a neighborhood of K and gluing it back in exactly the same way; this will always give S^3. Non-trivial surgery involves gluing a solid torus along any other slope.) This extends a theorem of Lackenby about alternating knots [17] to many non-alternating ones.

2.2. **Diagrammatic estimates on volume.** The twist number $\mathrm{tw}(D)$ of a diagram $D(K)$ often provides a good estimate on the volume of $S^3 \smallsetminus K$. Lackenby showed that the volume of a link is bounded above by a constant multiple of $\mathrm{tw}(D)$, where D is allowed to be *any* diagram of K [18]. For alternating diagrams, he also proved a lower bound in terms of $\mathrm{tw}(D)$. With improved constants, due to Agol, Storm, D. Thurston, and W. Thurston [1], the gap between the upper and lower bounds is less than a factor of 6. Kalfagianni, Purcell, and I proved a similarly tight two-sided bound for diagrams with at least eight crossings per twist region [12]:

$$1.80 \cdot (\mathrm{tw}(D) - 1) \ < \ \mathrm{vol}(S^3 \smallsetminus K) \ < \ 10.15 \cdot (\mathrm{tw}(D) - 1). \tag{2}$$

The proof of (2) uses the diagram $D(K)$ to construct a link L with particularly simple geometry. The original link K can be recovered by Dehn filling some components of L. The simple geometry of L allows for relatively easy diagrammatic estimates on its volume, as well as on the lengths of filling slopes needed to recover K. As a final step, we apply the volume formula (1) to gain an estimate on the volume of $S^3 \setminus K$.

Similar methods also work to prove diagrammatic bounds on the volumes of several other families of links: for example, closed 3–braids and banded sums of alternating links. These results are described in [11]. Other, more topological methods, also yield volume estimates for adequate knots and links ([10], see below).

2.3. **Hyperbolic volume and the Jones polynomial.** The volume conjecture, formulated by Kashaev, Murakami, and Murakami, states that the volume of a hyperbolic knot $K \subset S^3$ can be computed from certain asymptotics of the Jones polynomial and its relatives [21]. If true, this conjecture would open up a number of new connections between the fields of hyperbolic geometry and quantum topology. While the precise form of this conjecture has only been established for very few knots, one may reasonably hope to show a coarse version: that the volume of K is determined, up to a constant factor, by a few coefficients of the Jones polynomial. For alternating links, this was shown by Dasbach and Lin [7], building on work of Lackenby [18]. Recently, Kalfagianni and I have generalized this statement to the much larger family of adequate links.

The definition of an *adequate link* is tailored around certain nice properties of the Jones polynomial. For example, the span of the polynomial equals the crossing number of the link. An adequate diagram $D(K)$ determines a particular graph $G_A(D)$, from which the entire polynomial can be computed ([6], with Dasbach *et al*), while the two leading and two trailing coefficients are particularly easy to read off. In particular, the second coefficient β and next-to-last coefficient β' appear to have control over hyperbolic volume. Improving earlier estimates proved jointly with Purcell [12], Kalfagianni and I showed [10] that for most adequate links,

$$3.66\,(\max\{|\beta|, |\beta'|\} - 1) \;<\; \mathrm{vol}(S^3 \setminus K) \;<\; 20.3\,(|\beta| + |\beta'| - 1)\,. \tag{3}$$

The proof of this result illustrates potentially deep connections between the Jones polynomial and classical geometric topology. An adequate diagram $D(K)$ determines an incompressible surface with boundary along K, whose spine is the graph $G_A(D)$. The complement of this surface in S^3 decomposes along essential annuli into I–bundle pieces and hyperbolic pieces, which are called *guts*. Agol, Storm, and Thurston showed that the Euler characteristic of the guts gives a lower bound on hyperbolic volume [1]. What we proved is that this Euler characteristic can in fact be seen from the graph $G_A(D)$ itself, yielding a connection between volume and the Jones polynomial.

3. Angled triangulations

The results described so far have all been coarse in nature: certain combinatorial measurements of a knot or a 3–manifold determine its geometric type, and estimate its hyperbolic volume up to a bounded factor. By contrast, the method described in this section can often construct the full hyperbolic metric on a manifold from combinatorial data. At a minimum, the method of angled triangulations still provides a wealth of topological information for a very small cost.

3.1. **Constructing hyperbolic metrics.** A manifold M with torus boundary has a combinatorial description as a union of *ideal tetrahedra*. These are tetrahedra whose vertices have been removed; one can think of the vertices as lying on the boundary at infinity. To attempt to geometrize M, we can assign each tetrahedron a particular hyperbolic shape; these shapes are parametrized up to isometry by dihedral angles. In order to glue the tetrahedra coherently and get a complete hyperbolic structure on M, three conditions need to be satisfied:

(a) The dihedral angles around each edge must add up to 2π.
(b) There must not be any shearing of faces as we go around an edge.
(c) The ideal vertices must fit together in a way that keeps ∂M infinitely far away.

An *angle structure* on M is an assignment of dihedral angles to tetrahedra that only needs to satisfy condition (a). Because this condition is linear, an angle structure is a solution to a linear system of equations and inequalities (the inequalities keep the dihedral angles positive, and the tetrahedra convex). Thus the space of all angle structures is an open, bounded, convex polytope, making this type of structure relatively easy to obtain. By contrast, Mostow rigidity implies that a geometric gluing satisfying all of (a)–(c) must be unique.

By the work of Casson and Rivin, angle structures form a very useful stepping stone to finding a geometric structure. Every angle structure has an associated *volume*, namely the sum of the volumes of all the tetrahedra. This defines a smooth, concave function on the polytope of solutions. Casson and Rivin showed that if the volume function has a critical point in this polytope, the corresponding tetrahedron shapes satisfy conditions (a)–(c), and give a hyperbolic metric [23]. Thus to construct the hyperbolic metric, it suffices to show the open polytope contains a maximum of the volume function.

François Guéritaud and I have carried out this program for two families of manifolds with sufficiently tractable combinatorics [15]. Guéritaud worked out the case of punctured torus bundles (as in Section 1.2), which admit ideal triangulations closely related to the monodromy map. Following this lead, I extended the method to complements of two-bridge links. In addition to knowing how to build the hyperbolic metric, this method is very useful for estimates: because the complete structure comes from the *maximum* of the volume function, every angle structure gives a lower bound on hyperbolic volume. This led to particularly sharp volume bounds for both bundles and links.

The explicit construction of the hyperbolic structure on these link complements allows one to locate both finite and infinite geodesics in a projection diagram of the link. In particular, every crossing in a reduced diagram of K defines an arc isotopic to a hyperbolic geodesic. This resolves a conjecture posed by Thistlethwaite.

3.2. **Angle structures determine geometric type.** For general manifolds, constructing a hyperbolic metric via the Casson–Rivin program appears to be a difficult task. Remarkably, however, the mere presence of an angle structure already implies that such a metric must exist. Casson and Lackenby have shown that if a manifold M with torus boundary admits an angled triangulation, all topological obstructions to a hyperbolic metric will vanish [17]. Thus, by Thurston's hyperbolization theorem, M must be hyperbolic.

Guéritaud and I have extended this result in two directions. For manifolds with torus boundary, we showed that the same conclusion holds if M is built out of *angled blocks*, namely polyhedral pieces that need not be simply connected [8]. This is helpful because for many manifolds, decompositions into blocks arise more naturally than tetrahedra. We also used Dehn surgery techniques to extend the Casson–Lackenby result to closed manifolds. More precisely: if N is obtained by Dehn filling on a manifold M, and M admits an angled triangulation in which the dihedral angles fit together correctly around the surgery slope, then N must be hyperbolic. The extra condition about the surgery slope is once again a linear equation in the angles. Thus linear programming – for which there are rapid algorithms – can detect a closed hyperbolic manifold.

4. Future directions

The method of angled triangulations suggests fruitful opportunities for attacking a number of open problems. Here are a few.

4.1. **Effective geometrization of fibered manifolds.** The family of 3–manifolds that fiber over the circle exhibits particularly close connections between geometry and combinatorics. For example, Brock showed that the action of the monodromy map on a complex of pants decompositions coarsely determines the volume of the manifold [3]. For fibered manifolds with boundary, Saul Schleimer and I recently showed that the monodromy's action on the arc complex estimates the volume of a maximal cusp neighborhood of the boundary torus.

These coarse results provide hope of an effective way to construct the hyperbolic metrics on fibered manifolds, following Casson and Rivin's technique. One important first obstacle in this project is the choice of ideal triangulation. For punctured tori, the combinatorics of the monodromy map canonically determines a triangulation, but this may no longer hold true for larger fibers. A potential source of good triangulations comes from the world of Veech groups and translation surfaces. It would be interesting to see whether these triangulations admit angle structures, and whether the volume maximization method can promote the angle structures to a true geometric structure. This project on effective geometrization is the subject of a currently pending focused research group proposal to the NSF (joint with Jaco, Luo, Maher, Rubinstein, and Tillmann).

4.2. **Volume estimates from angled triangulations.** Given a manifold M and an ideal triangulation, the space of angle structures on this triangulation is an open, convex polytope P. Every point of P, as well as of its compact closure, has a natural associated volume. Casson conjectured that this volume always gives a lower bound on the hyperbolic volume of M. When the polytope contains a critical point of the volume function, this conjecture is automatically true, because the critical point – necessarily a global maximum – gives the complete hyperbolic structure. On the other hand, when the volume is maximized on the frontier of P, we only obtain a singular metric, in which the tetrahedra glue correctly along some edges but not others. A careful study of this singular metric could suggest a way to modify the triangulation, progressing closer to the true hyperbolic structure on M.

4.3. **The linearization of geometry.** The theorem that only hyperbolic manifolds can admit angle structures turns a hard, non-linear problem (gluing tetrahedra coherently to obtain a hyperbolic metric) into a linear problem. It is worth asking what other geometric problems can be similarly "linearized." One promising candidate is the detection of *boundary slopes* of surfaces: the question of which slopes on ∂M occur as boundary circles of incompressible surfaces in M. The Culler–Shalen theory of $SL(2, \mathbb{C})$ representation varieties frequently detects boundary slopes [5], but the algebraic geometry involved is difficult and non-linear. Feng Luo, Stephan Tillmann, and I have conjectured that a linear condition on dihedral angles should also detect boundary slopes. We are currently pursuing this conjecture. Among the potential consequences of this conjecture would be a classification of the hyperbolic manifolds that admit the greatest number of non-hyperbolic Dehn fillings.

4.4. **Unknotting tunnels and triangulations.** One important question in 3–dimensional topology is the search for connections between minimal–genus Heegaard splittings and reduced, *efficient* triangulations. This question is wide open, even in the relatively simple case where the manifold is the complement of a knot K and the splitting comes from an *unknotting tunnel*, namely an arc τ such that the complement of $K \cup \tau$ is a handlebody. In this case, Cho and McCullough [4] have shown that a simple iterative procedure uniquely constructs every unknotting tunnel of every knot that has one. Saul Schleimer and I are currently working on a project to realize the Cho–McCullough moves via changing ideal triangulations. We hope to assign angle structures to these triangulations, and use these angle structures to gain a glimpse into the geometry of the knot and its tunnel. In particular, we believe that this approach can lead to progress on Sakuma's conjecture that the unknotting tunnel τ is isotopic to a hyperbolic geodesic.

4.5. **Mom structures and triangulations.** In recent work, Gabai, Meyerhoff, and Milley have shown that every small-volume hyperbolic 3–manifold has a *Mom(2)* or *Mom(3)* structure, namely a handle decomposition with particularly simple combinatorics [14]. The construction of a *Mom* structure is geometric, and naturally relates to the Ford–Voronoi domain of a manifold with torus boundary. It would be interesting to see whether angled triangulations can be used to detect the presence of a *Mom(n)* structure for small n. If a connection of this sort exists, then angle structures can lead to upper bounds on both the geometric and topological complexity of a manifold.

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