Problems and riddles

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All problems are solvable using only undergraduate mathematics.
Problems were gathered from various sources.

Geometry

1. Alice and Bob are playing cat and mouse. Trying to escape Bob, Alice jumped into a circular pool. Bob is standing on the side of the pool with Alice next to him in the water. Bob cannot swim but he runs 4 times as fast as Alice swims. How can Alice escape the pool without being catched by Bob?

2. Can you cut a square into 8 triangles whose angles are all < 90 degrees?

3. Given the points of coordinates $(0,0)$ and $(0,1)$, how can you draw the point $(1,0)$ **using only a compass**?

4. A convex smooth room has all its walls covered with mirrors. A ray of light travels in the room (horizontally). Show that there is a closed periodic path with 3 sides for the ray of light.

5. If all points in the plane are colored in red, green or blue, show that there are two points at distance 1 with the same color.

6. If all points in $\mathbb{R}^3$ are colored in red, green or blue, show there is a color $c \in \{R, G, B\}$ such that

$$\{d(A,B)\ /A,B \in \mathbb{R}^2 \text{ are both colored by } c\} = [0,\infty)$$

7. Let S be a finite set a points in the plane, such that for any two points A and B in S there is a third point C in S that is on the line (AB). Show that all points in S are aligned.

8. You have a painting that you want to hang to the wall with a rope. On the wall are planted n nails. Can you hang the painting in such a way that if you remove any of the n nails, the painting falls?

9. Given a (non-flat) triangle ABC, how to construct with ruler and compass the point M such that $AM + BM + CM$ is minimal?

10. Let C be a convex set in the plane with positive area. Show that one can find two orthogonal lines that divide C into 4 regions of equal area.

Algebra

1. Let G be the quotient of the free group $\langle a, b, c, d, e, \ldots, x, y, z\rangle$ on 26 generators by the relations $w = w'$ if w and w' are two English words that are pronounced the same. Show that G is trivial.

2. Can you construct a collar with 2^n black or white pearls, such that any sequence of n black or white pearls can be found as consecutive perls in the collar?

3. Let P be a polynomial with non-negative integer coefficients and unknown degree. You want to guess the polynomial P and can ask successively for values of P. How many values do you need to ask to guess P?

4. n persons are seated around a table, and each of them has a recipient containing some amount of milk. Starting with Person 1, they share their current amount of milk in equal parts with all the others (including themself). After they all have shared their milks, each person has exactly the same amount of milk as at the beginning. How much milk does each of them have if Person n has 1 liter of milk?

5. 100 mathematicians used a body swapping machine many times randomly. They realized that the machine cannot switch again the minds of two bodies that used the machine previously. How can they, with the help of 2 extra mathematicians that didn't use the machine at all, all switch their minds back to their original body?

6. Given $A = (a_{ij})_{1 \leq i,j \leq n}$ and $B = (b_{ij})_{1 \leq i,j \leq n}$, let $C = A * B$ be the matrix of coefficients $c_{ij} = a_{ij} b_{ij}$.
What are the matrices A such that $\det(A * B) = 0$ for any matrix B?

Arithmetics

1. It is possible to multiply an integer a by 5 using only 3 additions:

$$b := a + a = 2a, \; c := b + b = 4a, \; d = c + a = 5a$$

What is the minimal number of additions needed to multiply an integer by 255?

2. The little Sami just celebrated his birthday! Sami's age this year is special, because it is just after a square and just before a cube. What is Sami's age?

3. What is $2017^{2017^{2017^{\cdot^{\cdot^{\cdot^{2017}}}}}} \pmod{11}$? (with 2017 exponentiations)

4. Using permutations of coordinates and the operation

$$(a, b, c) \longrightarrow (a, b, 2a + 2b - c)$$

can you go from the triple $(1, 3, 8)$ to the triple $(2, 5, 13)$?

5. Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $f(2) = 2$ and $f(mn) = f(m)f(n)$ whenever $gcd(n, m) = 1$. Show that f is the identity map.

6. Let p and q be distinct primes. Show there are only finitely many pairs of consecutive integers whose only prime divisors are p and q.

Game theory

1. Alice and Bob play the following game:
The game starts with $n \geq 12$ successive integers. Starting with Alice, each of them crosses one uncrossed integer during his turn. The game ends when there are only two integers a and b left, and Alice wins if $gcd(a, b) > 1$, Bob wins otherwise. Who has a winning strategy?

2. Alice and Bob play the following game:
Initially there are 3 piles of stones containing 2017, 13 and 2 stones. Starting with Alice, each of them can remove any positive number of stones from a single pile. The player that takes the last stone wins. Who has a winning strategy?

Set theory

1. Let $(S, <)$ be a set with a partial order. We assume that any stricly increasing map $f : S \to S$ is injective. Does $<$ have to be a total order?

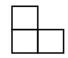

Figure 1: The triomino consists of 3 squares of dimension 1×1.

Combinatorics

1. On a chessboard, the legal moves of a knight are to move by 2 in one direction and by 1 in the other. The legal moves of a *steroid knight* consist of moving by 4 in one direction and by 5 in the other. On a 8×16 chessboard, can a steroid knight move from the corner $(0,0)$ to the corner $(7,0)$?

2. For which $(m,n) \in \mathbb{Z}^2$ can you cover the rectangle of dimension $n \times m$ with rectangles of dimensions $1 \times d$, without any overlap?
What about rectangles that you can cover with the triomino? (Figure 1) In both cases, you are allowed to rotate the pieces.

3. Three mirrors have been placed to form a equilateral triangle. Moreover, the corners of the cavity have been cut infinitesimally. How many ways is it possible for a ray of light to enter the cavity by one corner, reflect exactly 11 times and come out of the cavity by the same corner?
(Assume the ray of light comes in horizontally)

4. n people are sited around a table and want to clink their glasses. The clinking has to be done over the table. Each clink has to be between two persons only. During each round, any number of pairs of people may clink their glasses, but they must not cross their arms.
How many rounds are needed so that everyone can clink their glass with everyone else?

5. Now, what happens in the previous question if people are allowed to cross arms while clinking?

Riddles

1. You have a stock of 3000 bananas in town A and want to deliver as much as possible to town B which is 1000 miles away from town B. There is nothing but desert between these towns. To achieve your mission, you have a camel that can carry at most 1000 bananas at a time, and you can drop bananas on the road anywhere you like to pick them up later. However, your camel has to eat 1 banana per mile traveled.
How many bananas can you bring to town B?

2. Three explorers and three cannibals want to cross a river. They have a boat that can carry two people. All explorers know how to drive the boat, as well as one of the cannibals. However whenever the cannibals are a majority on either side of the river they will eat the explorers. Note also that one needs a few minutes to prepare the boat between two river crossings.
How can the explorers and cannibals all cross the river?

3. Uncle Scrooge found out that his purse of 100 coins contains a fake coin. His friend Gyro developed a machine to detect fake coins: given a pile of coins, the machine says whether the pile contains a fake coin or not. Gyro then charges Uncle Scrooge 2$ if his machine detected a fake coin in the pile or 1$ else. What is the least amount of dollars needed for Uncle Scrooge to detect the fake coin certainly?
(Assume Uncle Scrooge does not pay with the suspicious coins but from some other source)

4. You just bought the coolest watch! The hour and the minute needles are represented by two red lasers. You just realized however that this has two drawbacks: first it means that the two needles are undistinguishable. Moreover, because the lasers are very bright, if you take a look at your watch for an instant you have to wait a little while before looking at it again (in particular you cannot stare at your watch).
You know it's between 12:01pm and 11:59pm. If you look at your watch can you necessarily tell which time it is?

5. 100 mathematicians are being led by an evil logician to separate rooms containing each an infinite sequence of boxes labelled by $1, 2, 3, \ldots$. Communication between rooms is impossible. The evil logician has chosen a sequence u_i of real

numbers and put the number u_i in the box numbered i of each room. Each mathematician can open some (but not all) of the boxes in his room and then has to guess the number inside one of the box that he left closed. Note that the choice of boxes to open may depend on what he saw in previously opened boxes.
If more than one mathematician guesses wrong, they will all be executed. The mathematicians can decide of a strategy before entering their rooms. What should they do?

6. You are coming to a village to meet 3 brothers: Alex, Bob and Chris. You know that

   - Alex always lies.
   - Bob answers "Yes" or "No" randomly to any question.
   - Chris always tells the truth but hates trick questions. If you ask him a question with more than 4 words, he will punch you in the face.

   How can you to determine the identity of all three brothers, asking a maximum of 4 Yes/No question and without getting punched?
   (You can ask each of these questions to any of the brothers you like.)