MTH 411: Midterm exam 1
Fall 2016

Duration: 50 min
No calculator allowed

Exercise 1:

1) Find all elements in the cyclic subgroup $\langle 4 \rangle$ generated by 4 in U_{17}.
What is the index of $\langle 4 \rangle$ in U_{17}?

Correction: Elements in $\langle 4 \rangle$ are $\{1, 4, 4^2, \ldots\}$. We compute $4^2 \equiv 16$ [17], $4^3 \equiv 64 \equiv 13$ [17] and $4^4 \equiv 52 \equiv 1$ [17]. So 4 has order 4 in U_{17} and

$$\langle 4 \rangle = \{1, 4, 16, 13\}$$

The index of $\langle 4 \rangle$ in U_{17} is given by the formula $[U_{17} : \langle 4 \rangle] = \dfrac{|U_{17}|}{|\langle 4 \rangle|}$.
As 17 is prime, $U_{17} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ and $|U_{17}| = 16$.
Thus the index is $[U_{17} : \langle 4 \rangle] = \dfrac{16}{4} = 4$

2) Find all elements in the cyclic subgroup $\langle 10 \rangle$ generated by 10 in $\mathbb{Z}_{15}$.
Show that $\mathbb{Z}_{15}/\langle 10 \rangle \simeq \mathbb{Z}_5$

Correction: Elements in $\langle 10 \rangle$ are $\{0, 10, 2 \cdot 10, 3 \cdot 10 \ldots\}$. We compute $2 \cdot 10 \equiv 20 \equiv 5$ [15] and $3 \cdot 10 \equiv 30 \equiv 0$ [15]. Thus 10 has order 3 in $\mathbb{Z}_{15}$ and $\langle 10 \rangle = \{0, 5, 10\}$. The order of $\mathbb{Z}_{15}/\langle 10 \rangle$ is

$$|\mathbb{Z}_{15}/\langle 10 \rangle| = \frac{|\mathbb{Z}_{15}|}{|\langle 10 \rangle|} = \frac{15}{3} = 5$$

Because this order is a prime number, we know that $\mathbb{Z}_{15}/\langle 10 \rangle \simeq \mathbb{Z}_5$.

Exercise 2:

Let G be the set $\mathbb{R}^* \times \mathbb{R}$ and $\cdot$ be the operation on G defined by

$$(a, b) \cdot (c, d) = (ac, ad + b)$$

1) Show that $(G, *)$ is a group.

Correction: The operation is internal because $a \neq 0$ and $c \neq 0$ implies that $ac \neq 0$.
Moreover, for any $(a, b) \in G$, $(a, b) \cdot (1, 0) = (a, a \times 0 + b)$ and $(1, 0) \cdot (a, b) = (a, 1 \times b + 0) = (a, b)$. Thus $(1, 0)$ is an identity element for $\cdot$.
For $(a, b) \in G$, we have that

$$(a, b)(\frac{1}{a}, -\frac{b}{a}) = (\frac{a}{a}, -a\frac{b}{a} + b) = (1, 0)$$

and

$$(\frac{1}{a}, -\frac{b}{a})(a,b) = (\frac{a}{a}, \frac{b}{a} - \frac{b}{a}) = (1,0)$$

Thus $(\frac{1}{a}, -\frac{b}{a})$ is the inverse of (a,b).
Finally, for $(a,b),(c,d)$ and (e,f) in G, we have

$$(a,b)\left((c,d)(e,f)\right) = (a,b)(ce, cf+d) = (ace, acf+ad+b)$$

and

$$\left((a,b)(c,d)\right)(e,f) = (ac, ad+b)(e,f) = (ace, acf+ad+b)$$

Thus the operation is associative.

2) Let $H = \{(1,x) \ / \ x \in \mathbb{R}\}$.
Show that H is a normal subgroup of G.

Correction: First we note that $(1,0) \in H$. If $(1,x)$ and $(1,y)$ are two elements in H then

$$(1,x)(1,y) = (1, x+y) \in H$$

and

$$(1,x)^{-1} = (1,-x) \in H$$

So H is a subgroup of G.
For any $(a,b) \in G$ and $(1,x) \in H$ we have:

$$(a,b)(1,x)(a,b)^{-1} = (a,b)(1,x)(\frac{1}{a}, -\frac{b}{a}) = (a,b)(\frac{1}{a}, x - \frac{b}{a}) = (1, ax) \in H$$

Thus H is a normal subgroup of G.

Exercise 3:

Let G be a group of finite order and H and K be two subgroups of G.

1)Show that the intersection $H \cap K$ is a subgroup of G.

Correction: $e \in H$ and $e \in K$ so $e \in H \cap K$.
If x and y are in $H \cap K$ then $xy \in H$ and $x^{-1} \in H$ as H is a subgroup of G and $xy \in K$, $x^{-1} \in K$ as K is a subgroup of G. Thus $xy \in H \cap K$ and $x^{-1} \in H \cap K$. $H \cap K$ is thus a subgroup of G.

2) Using Lagrange's theorem, show that $|H \cap K|$ is a common divisor of $|H|$ and $|K|$.

Correction: By Lagrange theorem, if A is a subgroup of a finite group B then $|A|$

divides $|B|$. As $|H \cap K|$ is a subgroup of G, it is a group and thus a subgroup of both H and K. So the order $|H \cap K|$ divides the order of both subgroups: it is a common divisor of $|H|$ and $|K|$.

Exercise 4:

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 8 & 1 & 6 & 7 & 5 & 4 \end{pmatrix}$
Is σ^{411} even or odd?

Correction: $\sigma^{411} = (\sigma^{205})^2\sigma$. The square of any permutation is always even, so σ^{411} is even if and only if σ is even.
Now the cycle decomposition of σ is $\sigma = (12384)(567)$. It is the product of a 3-cycle and a 5-cycle, which are both even permutations, so σ is even and so is σ^{411}.