MTH 411: Correction exam 2
Fall 2015

Problem 1:

1) We have that $|G| = 105 = 3 \times 5 \times 7$. By the third Sylow theorem, the number of Sylow 5 subgroups is congruent to 1 modulo 5 and divides 21. The only possiblities are 1 and 21. Also, the number of Sylow 3 is congruent to 1 modulo 3 and divides 35, so it is either 1 or 7.
The number of Sylow 7 is congruent to 1 modulo 7 and divides 15, so it is either 1 or 15.

2) If S and S' are two distinct Sylow 5 subgroups of G, then they are both of cardinal 5, and $S \cap S'$ is a strict subgroup of S. So its cardinal is strictly less than 5 and divides 5 by Lagrange's theorem. Thus $S \cap S' = \{e\}$. The same reasoning applies for Sylow 7 subgroups of G, as their cardinals are 7, a prime number.

3) If G is simple then there is 21 Sylow 5 subgroups and 15 Sylow 7 subgroups. All Sylow subgroups consist of the identity element and 4 elements of order 5. As they have always trivial intersection, there must be at least $21 \times 4 = 84$ elements of order 5. By the same argument there is at least $15 \times 6 = 90$ elements of order 7. As $90 + 84 > 105$ we get a contradiction, so G is not simple.

Problem 2:

1) Write $x = a + b\sqrt{-13}$. Either $|b| \geq 1$, then $N(a + b\sqrt{-13}) = a^2 + 13b^2 \geq 13$.
Or $b = 0$ and $N(x) = a^2$. So the only norms less than 13 we can get are 0,1,4 or 9.

2) We have $N(2) = 4$, $N(11) = 121$, $N(3 + \sqrt{-13} = 22$ and $N(3 - \sqrt{-13}) = 22$. As $N(3 + \sqrt{-13}) = 22 \neq \pm N(2)$ and $\neq N(11)$, $3 + \sqrt{-13}$ is not an associate of 2 or 11. If 2 was not irreducible, any irreducible factor p of 2 should have norm $N(p)$ be a strict non-unit divisor of $N(2) = 4$, so we should have $N(p) = 2$. This is a contradiction as no element in R has norm 2. So 2 is irreducible. As no element has norm 11, we conclude also that 11, $3 + \sqrt{-13}$ and $3 - \sqrt{-13}$ are irreducibles.

3) We have found two decompositions of $22 = 2 \times 11 = (3 + \sqrt{-13})(3 - \sqrt{-13})$ into non-associate irreducible, so R is not a unique factorization domain.

Problem 3:

1) We show that I is an ideal. We have that $0 \in I$ as $0 \in I_n$ for any n. If i and j are two elements of I, then $i + j \in I_n$ for any n as I_n is an ideal. So $i + j \in I$. Also, if $\lambda \in R$, then $\lambda i \in I_n$ for any n as I_n is an ideal, so $\lambda i \in I$. As R is a principal ideal domain, there

exists $x \in R$ such that $I = (x)$.
As $(x) \subset I_n = (x_n)$, we have that $x = 0$ or x_n divides x.

2) If $I_n \not\supsetneq I_{n+1}$ then x_n is a strict divisor of x_{n+1}. So x_{n+1} must have at least one more irreducible factor than x_n, and x has an infinite number of irreducible factors, thus $x = 0$.