MTH 320: Midterm exam 2
Fall 2015

Duration: 50 min
The problems are independent.

Problem 1:
1) Let f_n be a sequence of continuous fonctions on $[0,1]$ converging uniformly to a function f. Let $x_n \in [0,1]$ be a sequence converging to $x \in [0,1]$.
Show that

$$f_n(x_n) \to f(x)$$

2) Let

$$g_n(x) = \frac{nx}{1+nx}$$

Show that $g_n(x) \to g(x)$, where g is the function on $[0,1]$ such that $g(x) = 1$ if $x > 0$ and $g(0) = 0$
3) What is $\lim_{n\to\infty} g_n(\frac{1}{n})$?

Problem 2:
Let $f : [0,1] \to [0,1]$ be a continuous function.
1) Applying the intermediate value theorem to the function $g(x) = f(x) - x$, show that $\exists c \in [0,1]$ such that $f(c) = c$.
A point $x \in [0,1]$ such that $f(x) = x$ is called a fixed point of f.
2) We now assume that f is differentiable on $[0,1]$ and $f'(x) \neq 1$ for any $x \in [0,1]$. Deduce from the mean value theorem that f has only one fixed point in $[0,1]$.

Problem 3:
1)Let $f_n : [0,\infty) \to \mathbb{R}$ be the function defined by

$$f_n(x) = \frac{e^{-n^2x}}{n^2}$$

Show that

$$F = \sum_{k=1}^{\infty} f_k$$

is a continuous function on $[0,\infty)$ and is differentiable on $[a,\infty)$ for any $a > 0$.
2) Show that F' is increasing over $(0,\infty)$ and that $F'(\frac{1}{n^2}) \leq -ne^{-1}$ for any $n \in \mathbb{N}$. Conclude that $F'(x)$ tends to $-\infty$ as $x \to 0$.