MTH 320: Correction midterm exam 2
Fall 2015

Problem 1:

1) By the triangle inequality, we have that

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \leq \sup_A |f_n - f| + |f(x_n) - f(x)|$$

The first term tends to zero as the convergence is uniform, the second tends to zero as f is continuous at x, as a uniform limit of continuous functions.

2) We have that $g_n(0) = \dfrac{n \times 0}{1 + n \times 0} = 0$ which tends to 0 as $n \to \infty$ and if $x > 0$ then

$$g_n(x) = \frac{nx}{1 + nx} = \frac{1}{1 + \frac{1}{nx}} \underset{n \to \infty}{\to} 1$$

3) $g_n(\dfrac{1}{n}) = \dfrac{n \times \frac{1}{n}}{1 + n \times \frac{1}{n}} = \dfrac{1}{2} \underset{n \to \infty}{\to} \dfrac{1}{2}$

Problem 2:

1) The function g is continuous on $[0, 1]$ and $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$ as f takes its values in $[0, 1]$.
By the Intermediate Value Theorem, there exists a point c in $[0, 1]$ such that $g(c) = 0$, that is $f(c) = c$.

2) Suppose $f(c) = c$ and $f(d) = d$ where $c < d$. Then as f is continuous and differentiable on $[c, d]$, by the Mean Value Theorem, $\exists x \in (c, d)$ such that

$$f'(x) = \frac{f(c) - f(d)}{c - d} = \frac{c - d}{c - d} = 1$$

Problem 3:

1)We have that $|f_n(x)| \leq \dfrac{1}{n^2}$ for any $x \in [0, \infty)$, so by the Weierstrass M-test, the serie converges uniformly on $[0, \infty)$ and the sum is a continuous function as the f_n are continuous on $[0, \infty)$.
For any n, the function f_n is differentiable on $[0, \infty)$ and $f_n'(x) = -e^{-n^2 x}$.
If $a > 0$ then $|f_n(x)| \leq e^{-n^2 a} \leq e^{-na}$ and $\sum_{n \in \mathbb{N}} e^{-na}$ is convergent, so by the Weierstrass M-test, the serie of f_n' is uniformly convergent on $[a, \infty)$ thus differentiable on $[a, \infty)$.

2) If $0 < x < y$ then $f_n'(x) = -e^{-n^2 x} < -e^{-n^2 y} = f_n'(y)$. Thus $F'(x) = \underset{n \in \mathbb{N}_n}{f\ '}(x) < \sum_{n \in \mathbb{N}} f_n'(y) = F'(y)$.

Finally, $F'(\frac{1}{n^2}) = \sum_{k\in\mathbb{N}} - e^{-\frac{k^2}{n^2}} \geq \sum_{k=1}^{n} - e^{-\frac{k^2}{n^2}} \leq \sum_{k=1}^{n} - e^{-\frac{n^2}{n^2}} = -ne^{-1}$.

So if $x < \frac{1}{n}$ then $F'(x) < -ne^{-1}$ as F' increasing, thus F' tends to $-\infty$ when $x \to 0$.