MTH 310: Final
Fall 2017

Duration: 120 min
No calculator allowed

Exercise 1:
Compute $2017^{2017} \mod 5$. (*Hint:* Show that $2017^4 = 1 \mod 5$ first.)

Exercise 2:
Show that $P(x) = x^4 + 6x^2 + 4$ is irreducible in $\mathbb{Q}[x]$.

Exercise 3:
In the ring $\mathbb{Q}[x]/(x^2+x+1)$, compute $[x]^k$ for $k = 0, 1, 2, 3, 4, 5$ and 6. Write your answer in the form $[ax+b]$ with a and $b \in \mathbb{Q}$.

Exercise 4:
a) Show that $F = \mathbb{Z}_5[x]/(x^3+3x+2)$ is a field.
b) How many elements are there in F?

Exercise 5:
Let

$$R = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid c = 0 \mod 3\}.$$

Show that R is a subring of $M_2(\mathbb{Z})$.

Exercise 6:
Let $I = \{P(x) \in \mathbb{R}[x] | P(0) = P'(0) = 0\}$.
a) Show that I is an ideal of $\mathbb{R}[x]$.
b) Is I a prime ideal?

Exercise 7:
Let $I = \{P(x) \in \mathbb{R}[x] \mid P(0) = P(2) = 0\}$.
a) Show that I is an ideal of $\mathbb{R}[x]$.
b) Show that $\mathbb{R}[x]/I \simeq \mathbb{R} \times \mathbb{R}$.

Problem:

1)Prove that $P(x) = x^2 + 1$ and $Q(x) = x^2 + 2x + 2$ are irreducible polynomials in $\mathbb{Z}_3[x]$.
2)Let $F = \mathbb{Z}_3[x]/(P(x))$. Prove that $[x-1] \in F$ is a root of $Q(x)$.
3)Let φ be the map

$$\begin{array}{cccc} \varphi & : & \mathbb{Z}_3[x] & \to & \mathbb{Z}_3[x]/(x^2+1) \\ & & R(x) & \to & R([x-1]) \end{array} .$$

Show that φ is a surjective morphism. (Hint: For surjectivity, compute $\varphi(ax+b)$ for a and $b \in \mathbb{R}$.)
4)Show that the kernel of φ contains the ideal $(Q(x))$.
5)Show that $(Q(x))$ is a maximal ideal of $\mathbb{Z}_3[x]$.
6)Deduce from 3),4) and 5) that the kernel of φ is exactly $(Q(x))$ and that

$$\mathbb{Z}_3[x]/(P(x)) \simeq \mathbb{Z}_3[x]/(Q(x)).$$