MTH 411: Midterm exam 2
Fall 2015

Duration: 50 min
The 3 problems are independent

Problem 1:
Let G be a group such that $|G| = 105$
1) Using the third Sylow theorem, show that there is either 1 or 21 Sylow 5 subgroups. Compute the possible numbers of Sylow 3, and Sylow 7 subgroups.
2) Show that the intersection of two distinct Sylow 5 subgroup of G is $\{e\}$. Does that apply to Sylow 7 subgroup?
3) We assume by contradiction that G is simple. Show that there must be at least 84 elements of order 5 and 90 elements of order 7 in G. Conclude that G is not simple.

Problem 2:
Let R be the integral domain $\mathbb{Z}[\sqrt{-13}]$. We recall that norm on R is the function $N : R \to \mathbb{Z}$ such that $N(a + b\sqrt{-13}) = a^2 + 13b^2$
1) Let $x \in R$. Show that either $N(x) \geq 13$ or $N(x) = 0, 1, 4$ or 9.
2) Show that 2, 11, $3 + \sqrt{-13}$ and $3 - \sqrt{-13}$ are irreducibles and that $3 + \sqrt{-13}$ is not an associate of 2 or 11.
3) Using the identity $22 = (3 + \sqrt{-13})(3 - \sqrt{-13})$, conclude that R is not a unique factorization domain.

Problem 3:
Let R be a principal ideal domain and $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ be a decreasing sequence of nonzero ideals.
We write $I_j = (x_j)$ with $x_j \neq 0$.
Let $I = \bigcap_{j \in \mathbb{N}} I_j$
1) Show that $\exists x \in R$ such that $I = (x)$ and that either $x = 0$ or for all $j \in \mathbb{N}$, x_j divides x.
2) We now assume that the sequence of ideals is strictly decreasing:

$$I_1 \not\supseteq I_2 \not\supseteq \ldots \not\supseteq I_n \not\supseteq \ldots$$

Show that $x = 0$.
(*Hint:* Compare the number of irreducible factors of x_i and x_{i+1}.)