MTH 320: Correction midterm exam 2
Fall 2015

Problem 1:

1)We have that $f_n'(x) = (-2nx+2)e^{-nx^2+2x-\frac{n^2+1}{n}}$ for any $x \in [0,1]$.
$f_n'(x) > 0 \Longleftrightarrow -2nx+2 > 0 \Longleftrightarrow x < \frac{1}{n}$.
Therefore, f_n is increasing over $[0, \frac{1}{n}]$ and decreasing over $[\frac{1}{n}, 1]$.

2) From the variation of f_n we get that $sup f_n = f_n(\frac{1}{n}) = e^{-n\cdot\frac{1}{n^2}+\frac{2}{n}-\frac{n^2+1}{n}} = e^{-n}$.
3)f_n is a positive function, so from 2) we get that $\sup\limits_{[0,1]}|f_n| = e^{-n} \underset{n\to\infty}{\longrightarrow} 0$. Thus f_n converges uniformly to 0 on $[0,1]$/

Problem 2:

1)For fixed $x \in [0,1]$, the sequence $\frac{1}{n+x}$ is positive, decreasing and tends to 0. Thus $\sum\limits_{n\in\mathbb{N}} \frac{(-1)^n}{n+x}$ converges by the alternating serie theorem.
2)We have that $|f_n(0)| = \frac{1}{n}$ and for any $x \in [0,1]$, $|f_n(x)| = \frac{1}{n+x} \leq \frac{1}{n}$. Thus $\sup\limits_{[0,1]} f_n = \frac{1}{n}$.
As the serie of general term $\frac{1}{n}$ does not converge, we cannot apply Weierstrass M-test to show uniform convergence.

3)

$$f_{2n}(x) + f_{2n+1(x)} = \frac{(-1)^{2n}}{2n+x} + \frac{(-1)^{2n+1}}{2n+1+x} = \frac{1}{2n+x} - \frac{1}{2n+1+x} = \frac{1}{(2n+x)(2n+1+x)}$$

Therefore, for any $x \in [0,1]$,

$$|f_{2n}(x) + f_{2n+1}(x)| \leq \frac{1}{4n^2}$$

4) $F_n = \sum\limits_{k=0}^{2n+1} f_k = \sum\limits_{k=0}^{n} f_{2k} + f_{2k+1}$ converges uniformly on $[0,1]$ by question 2) and the Weierstrass M-test. This is a subsequence of the sequence of partial sums of $\sum\limits_{n\in\mathbb{N}} \frac{(-1)^n}{n+x}$, thus it converges to f. The function f is continuous on $(0,1]$ as the uniform limit of the sequence of continuous functions F_n.

Problem 3:

1)We use the ratio test: for any $n \in \mathbb{N}$,

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)^{n+1}}{n^n} \geq (n+1) \cdot \frac{n^n}{n^n} \underset{n\to\infty}{\longrightarrow} \infty$$

Thus the radius of convergence is $R = \infty$.
2)As the radius of convergence is ∞, the serie of functions $g(x) = \sum_{n\in\mathbb{N}} \frac{x^n}{n^n}$ is uniformly convergent on $[0,1]$.
Thus g is integrable and $\int_0^1 g = \sum_{n\in\mathbb{N}} \int_0^1 \frac{x^n}{n^n} = \sum_{n\in\mathbb{N}} \frac{1}{(n+1)n^n}$

Problem 4:

1)

$$\int_0^1 f_n = \int_0^{\frac{1}{n}} nx + \int_{\frac{1}{n}}^{\frac{2}{n}} 2 - nx + \int_{\frac{2}{n}}^1 0 = [\frac{nx^2}{2}]_0^{\frac{1}{n}} + [2x - \frac{nx^2}{2}]_{\frac{1}{n}}^{\frac{2}{n}} = \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$$

2) First $f_n(0) = 0$ for any n, thus $f_n(0) \to 0$.
If $x > 0$, if $\frac{1}{N} \leq x$ then for any $n \geq N$ we have $f_n(x) = 0$. Thus $f_n(x) \underset{n\to\infty}{\longrightarrow} \infty$ 0.
If $\alpha > 0$, and $\frac{1}{N} < \alpha$ then for any $n \geq N$, $\sup_{[\alpha,1]}|f_n| = 0$ as f_n is identically 0 on $[\alpha, 1]$. Thus f_n converges uniformly to 0 on $[\alpha, 1]$.

3) Let $\alpha > 0$. We have that

$$|\int_0^1 g_n| \leq \int_0^\alpha |g_n| + \int_\alpha^1 |g_n| \leq M\alpha + \underset{[\alpha,1]}{sup}|g_n|$$

This last quantity will be less than $(M+1)\alpha$ if n is big enough, as g_n converges uniformly to 0 on $[\alpha, 1]$. Thus $\int_0^1 g_n \underset{n\to\infty}{\longrightarrow} 0$.