MTH 320: Final exam
Fall 2015

Duration: 120 min
The problems are independent.

Problem 1:
Let $f_n : [0,1] \longrightarrow \mathbb{R}$ be the function defined by

$$f_n(x) = e^{-nx^2+2x-\frac{n^2+1}{n}}$$

1)Compute the derivative of f_n and show that f_n is increasing on $[0, \frac{1}{n}]$ and decreasing on $[\frac{1}{n}, 1]$

2) Deduce from question 1 the value of $\sup\limits_{[0,1]} f_n$.

3)Conclude that $f_n \underset{n\to\infty}{\longrightarrow} 0$ uniformly on $[0,1]$.

Problem 2:

Let $f_n : [0,1] \longrightarrow \mathbb{R}$ be the function defined by

$$f_n(x) = \frac{(-1)^n}{n+x}$$

1)Using the Alternating Serie Theorem, show that for any $x \in [0,1]$, $\sum\limits_{n\in\mathbb{N}} f_n(x)$ converges.

We write $f(x) = \sum\limits_{n\in\mathbb{N}} f_n(x)$.

2)Show that $\sup\limits_{[0,1]} |f_n| = \frac{1}{n}$. Can we apply Weierstrass M-test to prove that the sum $\sum\limits_{n\in\mathbb{N}} f_n(x)$ is uniformly convergent?

3) Compute $f_{2n} + f_{2n+1}$ and show that for any $x \in [0,1]$, $|f_{2n}(x) + f_{2n+1}(x)| \leq \frac{1}{4n^2}$

4) Deduce that the sequence of functions $F_n = \sum\limits_{k=0}^{2n+1} f_k$ converges uniformly on $[0,1]$ and that f is continuous on $(0,1]$.

Problem 3:

1)What is the radius of convergence of the serie

$$g(x) = \sum_{n\in\mathbb{N}} \frac{x^n}{n^n}$$

2) Justifying your answer, express $\int_0^1 g$ as a serie of real numbers.

Problem 4:

Let $f_n : [0,1] \to \mathbb{R}$ be the function defined by

- $f_n(x) = 0$ if $x \in [\frac{2}{n}, 1]$
- $f_n(x) = nx$ if $x \in [0, \frac{1}{n}]$
- $f_n(x) = 2 - nx$ if $x \in [\frac{1}{n}, \frac{2}{n}]$.

1) Compute $\int_0^1 f_n$

2) Show that f_n converges to 0 pointwise on $[0,1]$ and that f_n converges to 0 uniformly on $[\alpha, 1]$ for any $\alpha > 0$.

3) Let $g_n : [0,1] \to \mathbb{R}$ such that
$g_n \to 0$ uniformly on $[\alpha, 1]$ for any $\alpha > 0$
and $\exists M$ such that $|g_n(x)| \leq M$ for any $x \in [0,1]$ and any $n \in \mathbb{N}$.
Show that $\int_0^1 g_n \to 0$.