MTH 320: Midterm exam 1
Fall 2015

Problem 1:

For any $n \in \mathbb{N}$, let $u_n = \dfrac{(-1)^n}{2n+3(-1)^n}$ and $v_n = \dfrac{(-1)^n}{2n}$

1) Show that the serie $\sum_{n\in\mathbb{N}} v_n$ converges conditionally but not absolutely.

We have $|v_n| = \dfrac{1}{2n}$, and we know that the harmonic serie $\sum \dfrac{1}{n}$ diverges. Thus this serie does converge absolutely. However, $\dfrac{1}{2n}$ is a positive decreasing sequence that converges to 0, so by the alternating serie theorem, $\sum_{n\in\mathbb{N}} v_n$ converges.

2) Explain why it is not possible to use the alternating series theorem to prove that $\sum_{n\in\mathbb{N}} u_n$ is convergent.

The sequence $b_n = \dfrac{1}{2n+3(-1)^n}$ is not decreasing as $b_{2n+1} = \dfrac{1}{4n-1} > \dfrac{1}{4n+3} = b_{2n}$. Thus we can not apply the alternating serie theorem.

3) Prove that $|u_n - v_n| \leq \dfrac{1}{2n^2}$ for any $n \geq 3$.

We have $|u_n - v_n| = \dfrac{|(-1)^n|}{2n|2n+3(-1)^n}| \leq \dfrac{1}{2n(2n-3)} \leq \dfrac{1}{2n^2}$ if $n \geq 3$, as then we have $2n - 3 \geq n$.

4) Conclude that the serie $\sum u_n$ is also convergent.

From the inequality above, we deduce that $\sum_{n\in\mathbb{N}} u_n - v_n$ converges absolutely by the comparison theorem. Thus, as $u_n = v_n + u_n - v_n$ is the sum of two sequences with convergent series, we can conclude that $\sum_{n\in\mathbb{N}} u_n$ is a convergent serie.

Problem 2:

Let a_n and b_n be defined by $a_1 = a$ and $b_1 = b$, where $0 < a < b$ and for any $n \in \mathbb{N}$

$$a_{n+1} = \sqrt{a_n b_n} \text{ and } b_{n+1} = \frac{a_n + b_n}{2}$$

1) Show by recursion that for any n, we have

$$a_n < a_{n+1} < b_{n+1} < b_n$$

For $n = 1$, we have that $0 < a < b$. Thus, $0 < \sqrt{a} < \sqrt{b}$ so that $a = 2 = \sqrt{ab} > b$. Moreover $b_2 = \frac{a+b}{2} < b$.

Finally, $b_2 - a_2 = \frac{a+b-2\sqrt{ab}}{2} = \frac{(\sqrt{b}-\sqrt{a})^2}{2} > 0$.

Now, if we assume

$$a_n < a_{n+1} < b_{n+1} < b_n$$

then $a_{n+2} = \sqrt{b_{n+1}a_{n+1}} > a_{n+1}$ and $b_{n+2} = \frac{a_{n+1}+b_{n+1}}{2} < b_{n+1}$. And again we have

$$b_{n+2} - a_{n+2} = \frac{a_{n+1}+b_{n+1}-2\sqrt{a_{n+1}b_{n+1}}}{2} = \frac{(\sqrt{b_{n+1}}-\sqrt{a_{n+1}})^2}{2} > 0$$

Thus the inequality is true for any $n \in \mathbb{N}$.

2) Show that a_n and b_n are convergent and that they have the same limit.

a_n is increasing and bounded above by b, and b_n is decreasing and bounded below by a, so by the monotone convergence theorem, there exists l and $l' \in \mathbb{R}$ such that $a_n \to l$ and $b_n \to l'$. From the equality $b_{n+1} = \frac{a_n+b_n}{2}$ we get as $n \to +\infty$ that $l' = \frac{l+l'}{2}$. Thus $l = l'$.

Problem 3:

Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. We introduce the sets

$$A_n = \{u_k, k > n\}$$

and call K the set

$$K = \{\lim_{n\to+\infty} u_{\varphi(n)} / \ u_{\varphi(n)} \text{ convergent subsequence of } u_n\}$$

1) What is K if $u_n = (-1)^n$?

If $u_{\varphi(n)}$ is a subsequence converging to L, then $|u_{\varphi(n)}| = 1 \to |L|$ as n tends to $+\infty$. So $|L| = 1$, that is L is either 1 or -1. The subsequence u_{2n} is constant equal to 1 and the subsequence u_{2n+1} is constant equal to -1, so we get that $K = \{-1, 1\}$.

2) Show that $K = \bigcap_{n\in\mathbb{N}} \overline{A_n}$. Deduce that K is a non-empty compact subset of $\mathbb{R}$.

For any convergent subsequence $u_{\varphi(n)}$, we have that $u_{\varphi(n)} \in \overline{A_k}$ as long as $n \geq k$. As $\overline{A_k}$ is closed, the limit is also in $\overline{A_k}$, and thus $K \subset \bigcap_{n\in\mathbb{N}} \overline{A_n}$.

Let $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$. As $x \in \overline{A_1}$ we know there is $u_{\varphi(1)}$ such that $|u_{\varphi(1)} - x| < \frac{1}{2^1}$.
As $x \in \overline{A_{\varphi(1)}}$ we know there is $\varphi(2) > \varphi(1)$ such that $|u_{\varphi(1)} - x| < \frac{1}{2^2}$.
We can recursively define an increasing sequence $\varphi(n)$ such that $|u_{\varphi(n)} - x| < \frac{1}{2^n}$, and thus $u_{\varphi(n)}$ converges to x and $x \in K$.

K is the intersection of closed sets $\overline{A_n}$ so it is closed. Furthermore there exists a bound B such that $|u_n \leq B$ for any n. For any convergent subsequence $u_{\varphi(n)}$, the limit L verifies also $|L| \leq B$ by the order limit theorem. Thus K is bounded. K is a closed bounded subset of $\mathbb{R}$, that is K is compact.

3) We now assume that $|u_n - u_{n+1}| \underset{n\to+\infty}{\longrightarrow} 0$. Show that K is connected.

As K is a subset of $\mathbb{R}$, we have to show that for any $a < c < b$ with a and b in K, c is also in K.
Let ε be a positive number, smaller than $|b - c|$. Let $N \in \mathbb{N}$ such that $|u_n - u_{n+1}| < \varepsilon$ as long as $n \geq N$.
For any $n > N$, $a \in \overline{A_n}$ so there is some $k > n$ such that $|a - u_k| < \varepsilon$
Moreover $b \in \overline{A_k}$ so there is some $l > k$ such that $|b - u_l| < \varepsilon$
Let $m = min\{k \leq n \leq l \ / \ u_n > c\}$. The minimum as exists as this is a non-empty set of integers as $u_l > b - \varepsilon > c$. Then

$$u_{m-1} \leq c \leq u_m$$

and $|u_{m-1} - u_m| < \varepsilon$. So $|u_m - c| < \varepsilon$. We have found elements of the sequence with arbitrary large rank which are ε-close to c, thus $c \in K$.