Ln-formally étale maps are not necessarily weakly étale

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The purpose of this note is to find an example that distinguishes between certain two weakened notions of étaleness of algebras (in the context of, typically, non-Noetherian rings), defined below.

Before recalling definitions of the two notions of étaleness, recall that an ideal $I \subseteq A$ in a commutative ring is called *locally nilpotent* if every element of I is nilpotent (but the nilpotency index can be, in general, unbounded).

Definition 0.1. Let $\varphi : R \to S$ be a homomorphism of commutative rings.

(1) φ is called *weakly étale* if both $\varphi : R \to S$ and $\mu : S \otimes_R S \to S$ are flat.

(2) φ is called *ln-formally étale* if it has the unique left lifting property with respect to all quotient maps $A \to A/I$, where I is locally nilpotent. That is, every commutative square of the form

$$\begin{array}{ccc} R & \xrightarrow{f} & A \\ \downarrow & {\scriptstyle \exists! h} \nearrow & \downarrow{\scriptstyle \mathrm{mod}\, I} \\ S & \xrightarrow{g} & A/I \end{array}$$

(with I locally nilpotent) admits the indicated factorization h, making both the resulting triangles commutative, and this factorization is unique.

If $\varphi : R \to S$ is weakly étale, by [1, Theorem 1.3 (3)], there is a faithfully flat extension $S \to T$ such that the composition $R \to T$ is a filtered colimit of étale R-algebras, hence ln-formally étale. By flat descent, one concludes that $R \to S$ is itself ln-formally étale.

The question on the converse statement was raised by Nikolaus.

Question 0.2. Is every ln-formally étale morphism weakly étale?

To answer the question in the negative, we produce an example of a ln-formally étale ring map which is not flat. The example is based on A. Geraschenko's example of formally smooth (actually formally étale) ring map which is not flat, [2].

Example 0.3. Let k be an arbitrary field (or any base). Consider the ring

$$R = k[{}_aX_w \mid a \in \mathbb{Z}_{\geq 1}, w \in (\mathbb{Z}_{\geq 0})^{<\omega}]/\mathfrak{Rel},$$

(i.e. the variables are indexed by positive integers a on the left and by finite words w in non-negative integers on the right), where

$$\mathfrak{Rel} = \left({}_{a+1}X_w{}^2 - {}_aX_w,\ \ {}_1X_{w0} \cdot {}_1X_{wa} - {}_aX_w \mid a \in \mathbb{Z}_{\geq 1}, w \in (\mathbb{Z}_{\geq 0})^{<\omega} \right) \ .$$

(That is, each variable ${}_aX_w$ has ${}_{a+1}X_w$ as square root, each collection of square roots $({}_aX_w)_{a\geq 1}$ has a "universal" common divisor ${}_1X_{w0}$, and the respective quotients are given by ${}_1X_{wa},\ a \geq 1$.)

Let $J \subseteq R$ be the ideal of all the variables, and consider the quotient map $\pi : R \to R/J = k$.

Proposition 0.4. *π is ln-formally étale.*

Proof. Consider a ring A with a locally nilpotent ideal I, and a commutative square

$$\begin{array}{ccc} R & \xrightarrow{f} & A \\ \downarrow & & \downarrow \\ k & \xrightarrow{g} & A/I \end{array} \ .$$

By the commutativity it is clear that $f(J) \subseteq I$. Obtaining a (necessarily unique) lift $h : k \to A$ amounts to showing that $f(J) = 0$.

To show that a variable ${}_aX_w$ is sent by f to 0, note that since $f({}_1X_{w0})$ is nilpotent, say $f({}_1X_{w0})^n = 0$, and ${}_1X_{w0}$ divides each ${}_{a+k}X_w$, we have

$$f({}_{a+k}X_w)^n = 0 \ \ \forall k \geq 0.$$

In particular, taking k large enough so that $2^k > n$, we have

$$f({}_aX_w) = f(({}_{a+k}X_w)^{2^k}) = f({}_{a+k}X_w)^{2^k} = f({}_{a+k}X_w)^{n+(2^k-n)} = 0.$$

This holds for every variable, so $f(J) = 0$, as desired. □

Proposition 0.5. *π is not flat. In particular, π is not weakly étale.*

Proof. Consider the short exact sequence

$$0 \longrightarrow ({}_1X_\emptyset) \longrightarrow R \longrightarrow R/({}_1X_\emptyset) \longrightarrow 0 \ .$$

($\emptyset$ being the empty word). Applying $- \otimes_R k (= - \otimes_R R/J)$, we obtain a complex

$$0 \longrightarrow ({}_1X_\emptyset)/J({}_1X_\emptyset) \longrightarrow k = k \longrightarrow 0, \ .$$

so to conclude non-flatness of π, it's enough to show that $J({}_1X_\emptyset) \subsetneq ({}_1X_\emptyset)$.

Assume for contradiction that $J({}_1X_\emptyset) = ({}_1X_\emptyset)$. so there is a relation of the form

$$(*) \qquad {}_1X_\emptyset = {}_1X_\emptyset f(\underline{X}) + rel(\underline{X}), \;\; f(\underline{0}) = 0, \; rel(\underline{X}) \in \mathfrak{Rel}$$

that holds in the full polynomial ring $k[{}_aX_w \mid a, w]$. Since it uses only finitely many variables, it holds also in the polynomial ring with finitely many variables

$$k[{}_aX_w \mid 1 \leq a \leq N, w \in \{0, 1, \ldots, N\}^{<N}]$$

for large enough N, and we thus deduce that in the "truncated version of R",

$$R_0 = k[{}_aX_w \mid 1 \leq a \leq N, w \in \{0, 1, \ldots, N\}^{<N}]/\mathfrak{Rel}_0,$$

$$\mathfrak{Rel}_0 = \left({}_{b+1}X_w{}^2 - {}_bX_w, \;\; {}_1X_{u0} \cdot {}_1X_{ua} - {}_aX_u \mid 1 \leq a, b+1 \leq N-1, w, u0, ua \in (\{0, 1, \ldots, N\})^{<N}\right),$$

the equality

$$J_0({}_1X_\emptyset) = ({}_1X_\emptyset)$$

also holds, where $J_0 \subseteq R_0$ denotes the ideal of variables. To conclude, we derive a contradiction with the last equality.

To that end, consider the map

$$\varphi : R_0 \to k[X^{2^{-\infty}}]$$

given by the following two recursive rules:

1. Assign ${}_0X_\emptyset \mapsto X$.
2. Whenever ${}_aX_w$ is assigned, and $a < N$ (so that ${}_{a+1}X_w$ is still a variable in R_0), recursively it follows that the value is of the form $\varphi({}_aX_w) = X^{b/2^k}$ for some integers $b, k > 0$. Then assign ${}_{a+1}X_w \mapsto X^{b/2^{k+1}} (= \sqrt{\varphi({}_aX_w)})$.
3. Whenever ${}_aX_w$ is assigned for a fixed word w (in letters $1, 2, \ldots, N$) of length $l < N-1$ and all $a = 1, 2, \ldots, N$, from the recursion it follows that ${}_NX_w$ was assigned to an element of the form $\varphi({}_NX_w) = X^{b/2^k}$ for some for some integers $b, k > 0$. Then assign ${}_1X_{w0} \mapsto X^{b/2^{k+1}} (= \sqrt{\varphi({}_NX_w)})$, and assign ${}_1X_{wa}$, $a = 1, 2, \ldots, N$ to the appropriate quotients, i.e. ${}_1X_{wa} \mapsto X^{(2^{N-a+1}-1)b/2^{k+1}} (= \varphi({}_aX_w)/\varphi({}_1X_{w0}))$.

The above procedure assigns values to all variables of R_0 in a manner compatible with the relations $\mathfrak{Rel}_0$. The result is a ring homomorphism

$$\varphi : R_0 \twoheadrightarrow k[X^{2^{-K}}]$$

for some $K \in \mathbb{Z}_{\geq 0}$, taking the ideal J_0 to $(X^{2^{-K}})$ and ${}_1X_\emptyset$ to X. From this we see that $\varphi(J_0({}_1X_\emptyset)) = (X^{1+2^{-K}}) \subsetneq (X) = \varphi(({}_1X_\emptyset))$, contradicting the equality $J_0({}_1X_\emptyset) = ({}_1X_\emptyset)$. □

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References

[1] Bhatt, B. and Scholze, P., *The pro-étale topology for schemes*. Astérisque, 369 (2015), 99–201.

[2] Geraschenko, A., *Is there an example of a formally smooth morphism which is not smooth?*. MathOverflow, `https://mathoverflow.net/q/200` .