π-typical Witt vectors

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The purpose of this note is to supply detailed proofs of existence and basic properties of "π-typical Witt vectors", that is, an analogue of p-typical Witt vectors when the prime p is replaced by a uniformizer π of a local number field, as introduced in [1], and further described in [2].

The basic setup is as follows. Let $E/\mathbb{Q}_p$ be a finite extension, $[E : \mathbb{Q}_p] = ef$ the ramification and inertia indices, let $\mathcal{O}_E$ be the ring of integers, $\pi \in \mathcal{O}_E$ a uniformizer, and $\kappa_E \simeq \mathbb{F}_q$ the residue field where $q = p^f$.

Denote by

$$U : \mathcal{O}_E{-}\mathsf{Alg} \longrightarrow \mathsf{Set} \text{ the forgetful functor,}$$

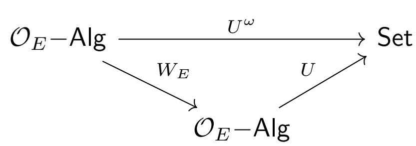
$$U^\omega : \mathcal{O}_E{-}\mathsf{Alg} \longrightarrow \mathsf{Set} \text{ the functor} A \mapsto A^\omega,$$

$(w_{\pi,n} =)w_n : U^\omega \Longrightarrow U$ the natural transformation given by

$$w_{n,A} : A^\omega \to A, (a_k)_{k\geq 0} \mapsto \sum_{k=0}^{n} \pi^k a_k^{q^{n-k}}$$

(and identify w_n with the polynomial $w_n(\underline{X}) = \sum_{k=0}^{n} \pi^k X_k^{q^{n-k}}$). The collection of all w_n's assemble to a natural transformation $(w_\pi =)w : U^\omega \Longrightarrow U^\omega$. Denote by $\mathrm{Id}^\omega : \mathcal{O}_E{-}\mathsf{Alg} \longrightarrow \mathcal{O}_E{-}\mathsf{Alg}$ the functor $A \mapsto A^\omega$ (with the product ring structure on A^ω).

Proposition 1. *There exists a unique functor $W_{E,\pi} = W_E : \mathcal{O}_E{-}\mathsf{Alg} \to \mathcal{O}_E{-}\mathsf{Alg}$ fitting into the commutative diagram*



such that $w : U^\omega \Longrightarrow U^\omega$ is a natural transformation $w : W_E \Longrightarrow \mathrm{Id}^\omega$.

Remark 2. When one takes $E = \mathbb{Q}_p$ and $\pi = p$, W_E recovers the standard p-typical vectors.

Lemma 3. *Given a polynomial $\Phi(X, Y) \in \mathcal{O}_E[X, Y]$, there exist polynomials $\Phi_n(\underline{X}, \underline{Y}) \in \mathcal{O}_E[X_0, X_1, \ldots X_n, Y_0, Y_1, \ldots, Y_n]$ such that $\forall n \geq 0$, $\Phi(w_n(\underline{X}), w_n(\underline{Y})) = w_n(\underline{\Phi}(\underline{X}, \underline{Y}))$, i.e.*

$$\Phi\left(\sum_{k=0}^{n} \pi^k X_k^{q^{n-k}}, \sum_{k=0}^{n} \pi^k Y_k^{q^{n-k}}\right) = \sum_{k=0}^{n} \pi^k \Phi_k(\underline{X}, \underline{Y})^{q^{n-k}}.$$

Proof. The polynomials $\phi_n(\underline{X},\underline{Y}) \in E[\underline{X},\underline{Y}]$ do exist and are necessarily unique. Indeed, clearly $\Phi_0(X,Y) = \Phi(X,Y)$, and $\Phi_n(\underline{X},\underline{Y})$ is obtained from $\Phi_0, \Phi_1, \dots, \Phi_{n-1}$ by

$$\Phi_n(\underline{X},\underline{Y}) = \frac{1}{\pi^n}\left(\Phi\left(\sum_{k=0}^{n}\pi^k X_k^{q^{n-k}}, \sum_{k=0}^{n}\pi^k Y_k^{q^{n-k}}\right) - \sum_{k=0}^{n-1}\pi^k \Phi_k(\underline{X},\underline{Y})^{q^{n-k}}\right) .$$

What remains is to show integrality of the coefficients of each Φ_n.

We proceed by induction. Suppose that all the polynomials $\Phi_0, \Phi_1, \dots, \Phi_{n-1}$ have integral coefficients. Let $\equiv$ denote the congruence mod π^n. Then

$$\sum_{k=0}^{n}\pi^k X_k^{q^{n-k}} = \sum_{k=0}^{n-1}\pi^k X_k^{q^{n-k}} + \pi^n X_n \equiv \sum_{k=0}^{n-1}\pi^k X_k^{q^{n-k}},$$

hence

$$\Phi\left(\sum_{k=0}^{n}\pi^k X_k^{q^{n-k}}, \sum_{k=0}^{n}\pi^k Y_k^{q^{n-k}}\right) \equiv \Phi\left(\sum_{k=0}^{n-1}\pi^k X_k^{q^{n-k}}, \sum_{k=0}^{n-1}\pi^k Y_k^{q^{n-k}}\right)$$

$$= \Phi\left(\sum_{k=0}^{n-1}\pi^k (X_k^{q})^{q^{n-1-k}}, \sum_{k=0}^{n-1}\pi^k (Y_k^{q})^{q^{n-1-k}}\right) = \sum_{k=0}^{n-1}\pi^k \Phi_k(\underline{X}^q,\underline{Y}^q)^{q^{n-1-k}} .$$

Since the polynomials Φ_k in the last expression are all integral by induction hypothesis, we have that $\Phi_k(\underline{X}^q,\underline{Y}^q)$ is congruent to $\Phi_k(\underline{X},\underline{Y})^q$ modulo π, i.e.

$$\Phi_k(\underline{X}^q,\underline{Y}^q) = \Phi_k(\underline{X},\underline{Y})^q + \pi A_k, \quad A_k \in \mathcal{O}_E[\underline{X},\underline{Y}] .$$

Thus, we obtain

$$\Phi\left(\sum_{k=0}^{n}\pi^k X_k^{q^{n-k}}, \sum_{k=0}^{n}\pi^k Y_k^{q^{n-k}}\right) \equiv \sum_{k=0}^{n-1}\pi^k \left(\Phi_k(\underline{X},\underline{Y})^q + \pi A_k\right)^{q^{n-1-k}} .$$

Now by binomial theorem, we have

$$\pi^k \left(\Phi_k(\underline{X},\underline{Y})^q + \pi A_k\right)^{q^{n-1-k}} = \pi^k \Phi_k(\underline{X},\underline{Y})^{q^{n-k}} + \sum_{j=1}^{q^{n-1-k}} \binom{q^{n-1-k}}{j}\pi^{j+k} B_k, \quad B_k \in \mathcal{O}_E[\underline{X},\underline{Y}],$$

and

$$\mathrm{val}_p\left(\binom{q^{n-1-k}}{j}\right) = f(n-1-k) - \mathrm{val}_p(j) \geq n-1-k-(j-1) = n-k-j,$$

hence

$$\mathrm{val}_\pi\left(\binom{q^{n-1-k}}{j}\pi^{j+k}\right) \geq e(n-k-j)+j+k \geq (n-k-j)+j+k = n.$$

Thus, we have that $\left(\Phi_k(\underline{X},\underline{Y})^q + \pi A_k\right)^{q^{n-1-k}} \equiv \Phi_k(\underline{X},\underline{Y})^q$, and we conclude that

$$\Phi\left(\sum_{k=0}^{n}\pi^k X_k^{q^{n-k}}, \sum_{k=0}^{n}\pi^k Y_k^{q^{n-k}}\right) \equiv \sum_{k=0}^{n-1}\pi^k \left(\Phi_k(\underline{X},\underline{Y})^q\right)^{q^{n-1-k}} = \sum_{k=0}^{n-1}\pi^k \Phi_k(\underline{X},\underline{Y})^{q^{n-k}},$$

as desired. $\square$

Lemma 4. *Let $x \in \mathcal{O}_E$. Then there are unique elements $s_n(x) \in \mathcal{O}_E$ with the property*

$$(w_n(\underline{s}(x)) =) \sum_{k=0}^{n} \pi^k s_k(x)^{q^{n-k}} = x \ \ \forall n.$$

Moreover, if Φ, Φ_n are as in Lemma 3, then we have

$$\forall x, y \in \mathcal{O}_E \ \forall n \geq 0 : \quad s_n(\Phi(x,y)) = \Phi_n(\underline{s}(x), \underline{s}(y)) \ .$$

Proof. For $x \in \mathcal{O}_E$, the elements $s_n(x) \in \mathcal{O}_X$ necessarily have to be given recursively as follows:

$$s_0(x) = x,$$

$$s_n(x) = \frac{1}{\pi^n} \left(x - \sum_{k=0}^{n-1} \pi^k s_k(x)^{q^{n-k}} \right) \ .$$

The claim is that these are well-defined elements of $\mathcal{O}_E$, which again boils down to inductively verifying the congruence

$$x \equiv \sum_{k=0}^{n-1} \pi^k s_k(x)^{q^{n-k}} \pmod{\pi^n}.$$

For $n = 1$, this is clear since $x \equiv x^q \pmod{\pi}$ for all $x \in \mathcal{O}_E$. To check it for a general n, note that we have

$$(*) := \sum_{k=0}^{n-1} \pi^k s_k(x)^{q^{n-k}} = \sum_{k=0}^{n-1} \pi^k (s_k(x)^q)^{q^{n-1-k}},$$

where $s_k(x)^q \equiv s_k(x) \pmod{\pi}$, i.e. $s_k(x)^q = s_k(x) + \pi t$, $t \in \mathcal{O}_E$, and after binomial expansion of $(s_k(x) + \pi t)^{q^{n-1-k}}$, the same estimates as in the proof of Lemma 3 show that

$$\pi^k (s_k(x) + \pi t)^{q^{n-1-k}} \equiv \pi^k (s_k(x))^{q^{n-1-k}} \pmod{\pi^n} \ .$$

Thus, we obtain

$$(*) \equiv \sum_{k=0}^{n-1} \pi^k s_k(x)^{q^{n-1-k}} = x \pmod{\pi^n} \ .$$

What remains is to verify the last identity, which holds for $n = 0$ and follows immediately by induction from the etablished relations:

$$s_n(\Phi(x,y)) = \frac{1}{\pi^n} \left(\Phi(x,y) - \sum_{k=0}^{n-1} \pi^k s_k(\Phi(x,y))^{q^{n-k}} \right) =$$

$$= \frac{1}{\pi^n} \left(\Phi \left(\sum_{k=0}^{n} \pi^k s_k(x)^{q^{n-k}}, \sum_{k=0}^{n} \pi^k s_k(y)^{q^{n-k}} \right) - \sum_{k=0}^{n-1} \pi^k \Phi_k(\underline{s}(x), \underline{s}(y))^{q^{n-k}} \right) = \Phi_n(\underline{s}(x), \underline{s}(y)) \ .$$

□

Proof of Proposition 1. To describe W_E on an object $A \in \mathcal{O}_E{-}\mathsf{Alg}$ is to endow A^ω with an $\mathcal{O}_E$-algebra structure that makes $w_A : A^\omega \to A^\omega$ a map of $\mathcal{O}_E$-algebras (where the ring structure on the codomain is component-wise).

To specify the ring structure, consider $\Phi_n(\underline{X}, \underline{Y})$ obtained from $\Phi(X, Y) = X + Y$, and $\Psi_n(\underline{X}, \underline{Y})$ obtained from $\Psi(X, Y) = XY$ by Lemma 3. Then in $W_E(A)$, set

$$(a_n)_n + (b_n)_n := (\Phi_n(\underline{a}, \underline{b}))_n,$$

$$(a_n)_n \cdot (b_n)_n := (\Psi_n(\underline{a}, \underline{b}))_n.$$

Finally, set $\underline{1} = (1, 0, 0, 0, \dots)$ and $\underline{0} = (0, 0, 0, \dots)$.

Note that the above ring structure has functoriality in the sense that given an $\mathcal{O}_E$-algebra map $f : A \to B$, the induced map $W_E(f) = f^\omega : W_E(A) \to W_E(B)$ (defined by f on each component) respects the above operations, so it is a morphism of algebraic structures with ring signatures, and it will be a ring homomorphism once we show that $W_E(A), W_E(B)$ are rings.

It is also clear from Lemma 3 (and the def. of w) that $w : W_E(A) \to A^\omega$ is a homomorphism of algebraic structures with ring signatures.

If A is a free $\mathcal{O}_E$-algebra, then we have $A \subseteq B := A[1/\pi]$, and it is easy to see that $w_B : W_E(B) \to B^\omega$ is an isomorphism, with inverse given as follows:

$$(w^{-1})_0(\underline{X}) := X_0,$$

$$(w^{-1})_n(\underline{X}) := \frac{1}{\pi^n}\left(X_n - \sum_{k=0}^{n-1} \pi^k (w^{-1})_k(\underline{X})^{q^{n-k}}\right).$$

Thus, it follows in this case that $W_E(B)$ is a commutative ring. Since $W_E(A) \subseteq W_E(B)$ is a substructure in the ring signature, $W_E(A)$ is a commutative ring as well.

Furthermore, note that checking for an $\mathcal{O}_E$-algebra A whether $W_E(A)$ with the operations defined above is a commutative ring amounts to checking whether certain polynomial identities (for Φ_n's and Ψ_n's) hold on A. But by the arguments above, the very same polynomial identities hold on all free $\mathcal{O}_E$-algebras, so on any $\mathcal{O}_E$-algebra A. Thus, we conclude that $W_E(A)$ is a commutative ring for all $\mathcal{O}_E$-algebras A.

Finally, we specify the $\mathcal{O}_E$-algebra structure on $W_E(A)$. For $A = \mathcal{O}_E$, we have a map $\underline{s} : \mathcal{O}_E \to W_E(A)$, which is a ring homomorphism by Lemma 4 (and the fact that $\underline{s}(1) = \underline{1}$). This gives an $\mathcal{O}_E$-algebra structure to $W_E(\mathcal{O}_E)$, and $W_E(A)$ of an arbitrary $\mathcal{O}_E$-algebra $\mathcal{O}_E \overset{\varphi}{\to} A$ receives an $\mathcal{O}_E$-algebra structure by

$$\mathcal{O}_E \xrightarrow{\underline{s}} W_E(\mathcal{O}_E) \xrightarrow{W_E(\varphi)} W_E(A).$$

This turns tautologically all maps $W_E(f) : W_E(A) \to W_E(B)$ and all $w_A : W_E(A) \to A^\omega$ into $\mathcal{O}_E$-algebra homomorphisms. □

Proposition 5. *In the above setup,*

(1) The functor $W_{E,\pi}$ is unique.

(2) Given a second uniformizer π', there is a natural isomorphism $\alpha_{\pi',\pi} : W_{E,\pi} \Longrightarrow W_{E,\pi}$ that is also natural in π, i.e. there is a functor from the groupoid (setoid) of uniformizers of E to $\mathsf{Fun}(\mathcal{O}_E{-}\mathsf{Alg}, \mathcal{O}_E{-}\mathsf{Alg})$, sending π to $W_{E,\pi}$ and $u = \pi'/\pi : \pi \to \pi'$ to $\alpha_{\pi',\pi}$.

Lemma 6. *Let π, π' be two uniformizers of E. There is a unique family of polynomials $\alpha_n(\underline{X}) = \alpha_{\pi',\pi,n}(\underline{X}) \in \mathcal{O}_E[X_0, \ldots, X_n]$, $n \geq 0$, such that for all n,*

$$\sum_{k=0}^{n} (\pi')^k \alpha_k(\underline{X})^{q^{n-k}} = \sum_{k=0}^{n} \pi^k X_k^{q^{n-k}}.$$

Additionally, the polynomials α satisfy

$$\alpha_{\pi,\pi,n}(\underline{X}) = X_0 \;\; \forall n,$$

$$\alpha_{\pi'',\pi',n}(\underline{\alpha_{\pi',\pi}}(\underline{X})) = \alpha_{\pi'',\pi,n}(\underline{X}) \;\; \forall n.$$

Moreover, if $\Phi_{\pi,n}(\underline{X}, \underline{Y})$, $\Phi_{\pi',n}(\underline{X}, \underline{Y})$ are polynomials as in Lemma 3 and for $x \in \mathcal{O}_E$, $s_{\pi,n}(x)$ and $s_{\pi',n}(x)$ are elements as in Lemma 4, then we have

$$\Phi_{\pi',n}(\underline{\alpha}(\underline{X}), \underline{\alpha}(\underline{Y})) = \alpha_n(\underline{\Phi_{\pi}}(\underline{X}, \underline{Y}))$$

and

$$\alpha_n(\underline{s_{\pi}}(x)) = s_{\pi',n}(x).$$

Proof. As in Lemmas 3 and 4, we necessarily have $\alpha_0(X_0) = X_0$ and

$$\alpha_n(\underline{X}) = \frac{1}{(\pi')^n} \left(\sum_{k=0}^{n} \pi^k X_k^{q^{n-k}} - \sum_{k=0}^{n-1} (\pi')^k \alpha_k(\underline{X})^{q^{n-k}} \right),$$

so we again only need to check

$$\sum_{k=0}^{n} \pi^k X_k^{q^{n-k}} \equiv \sum_{k=0}^{n-1} (\pi')^k \alpha_k(\underline{X})^{q^{n-k}} \pmod{(\pi')^n}$$

by induction on n. We have

$$\sum_{k=0}^{n-1} (\pi')^k \alpha_k(\underline{X})^{q^{n-k}} = \sum_{k=0}^{n-1} (\pi')^k (\alpha_k(\underline{X})^q)^{q^{n-1-k}} = \sum_{k=0}^{n-1} (\pi')^k (\alpha_k(\underline{X}^q) + \pi' t_k)^{q^{n-1-k}}, \; t_k \in \mathcal{O}_E,$$

and again the binomial expansion of $(\alpha_k(\underline{X}^q) + \pi' t_k)^{q^{n-1-k}}$ and the usual estimates thus show that

$$\sum_{k=0}^{n-1} (\pi')^k \alpha_k(\underline{X})^{q^{n-k}} \equiv \sum_{k=0}^{n-1} (\pi')^k (\alpha_k(\underline{X}^q) + \pi' t_k)^{q^{n-1-k}} \equiv \sum_{k=0}^{n-1} (\pi')^k (\alpha_k(\underline{X}^q))^{q^{n-1-k}} \pmod{(\pi')^n}.$$

By the induction hypothesis, the right-hand side equals

$$\sum_{k=0}^{n-1} \pi^k (X_k^q)^{q^{n-1-k}} = \sum_{k=0}^{n-1} \pi^k (X_k)^{q^{n-k}} \equiv \sum_{k=0}^{n} \pi^k (X_k)^{q^{n-k}} \pmod{(\pi')^n},$$

so we are done.

The identity $\alpha_{\pi,\pi,n}(\underline{X}) = X_n$ follows from the uniqueness of $\alpha_{\pi,\pi,n}(\underline{X})$ since $\alpha_n(\underline{X}) = X_n$ obviously works. Similarly, the identity $\alpha_{\pi'',\pi',n}(\underline{\alpha_{\pi',\pi}}(\underline{X})) = \alpha_{\pi'',\pi,n}(\underline{X})$ follows from uniqueness since

$$w_{\pi'',n}(\underline{\alpha_{\pi'',\pi'}}(\underline{\alpha_{\pi',\pi}}(\underline{X}))) = w_{\pi',n}(\underline{\alpha_{\pi',\pi}}(\underline{X})) = w_{\pi,n}(\underline{X}).$$

The additional identities follows from uniqueness of $\Phi_{\pi',n}(\underline{\alpha}(\underline{X}), \underline{\alpha}(\underline{Y}))$, $s_{\pi',n}(x)$ with respect to their defining properties together with the identities

$$\begin{aligned} w_{\pi',n}(\underline{\alpha}(\underline{\Phi_\pi}(\underline{X},\underline{Y}))) =& w_{\pi,n}(\underline{\Phi_\pi}(\underline{X},\underline{Y})) = \Phi(w_{\pi,n}(\underline{X})), w_{\pi,n}(\underline{Y})) \\ =& \Phi(w_{\pi',n}(\underline{\alpha}(\underline{X})), w_{\pi',n}(\underline{\alpha}(\underline{Y}))) \quad \Big(= w_{\pi',n}(\underline{\Phi_{\pi'}}(\underline{\alpha}(\underline{X}), \underline{\alpha}(\underline{Y})))\Big) \end{aligned}$$

and

$$x = w_{\pi,n}(\underline{s_\pi}(x)) = w_{\pi',n}(\underline{\alpha}(\underline{s_\pi}(x))).$$

□

Proof of Proposition 5. Since $w_{A,\pi}$ is injective for all subalgebras of algebras of the form $\mathcal{O}_E[\{X_i\}_i][1/\pi]$, the algebra structure $W_E(A)$ is uniquely determined on these algebras, in particular on free $\mathcal{O}_E$-algebras. Given a general $\mathcal{O}_E$-algebra A, let $f : B \to A$ be a sujective $\mathcal{O}_E$-algebra homomorphism. Then the functor U^ω determines that $W_{E,\pi}(f)$ is necessarily $f^\omega : W_{E,\pi}(B) \to W_{E,\pi}(A)$, and this is obviously surjective. Thus, to make this a ring homomorphism there is only one choice of a ring structure on $W_{E,\pi}(A)$. This proves (1).

To prove (2), set $\alpha_{\pi',\pi} W_{E,\pi} \Rightarrow W_{E,\pi'}$ to be the map polynomially given by polynomials from Lemma 6, that is,

$$\begin{aligned} \alpha_{\pi',\pi,A} : W_{E,\pi}(A) &\longrightarrow W_{E,\pi'}(A) \\ (a_k)_{k\geq 0} &\longmapsto (\alpha_k(\{a_l\}_{l\geq k}))_{k\geq 0} \quad . \end{aligned}$$

Then the relations with Φ_n's and s_n's from Lemma 6 immediately translate to the fact that each $\alpha_{\pi',\pi,A}$ is an $\mathcal{O}_E$-algebra homomorphism. Furthermore, Lemma 6 implies that $\alpha_{\pi'',\pi'} \circ \alpha_{\pi',\pi} = \alpha_{\pi'',\pi}$, proving the claimed functoriality as well as the fact that $\alpha_{\pi',\pi}$ is invertible with the inverse given by $\alpha_{\pi,\pi'}$. □

References

[1] V. G. Drinfel'd, *Coverings of p-adic symmetric domains,* Functional Analysis and its Applications **10** (1976), no. 2, p. 29–40.

[2] L. Fargues and J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique.* `https://webusers.imj-prg.fr/~laurent.fargues/Courbe_fichier_principal.pdf`