RAMIFICATION BOUNDS VIA WACH MODULES AND q–CRYSTALLINE COHOMOLOGY

PAVEL ČOUPEK

ABSTRACT. Let K be an absolutely unramified p–adic field. We establish a ramification bound, depending only on the given prime p and an integer i, for mod p Galois representations associated with Wach modules of height at most i. Using an instance of q–crystalline cohomology (in its prismatic form), we thus obtain improved bounds on the ramification of $\mathrm{H}^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})$ for a smooth proper p–adic formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$, for arbitrarily large degree i.

CONTENTS

1. INTRODUCTION

Let $p > 0$ be a prime and let K be a p–adic field. The aim of this note is to study properties of mod p G_K–representations T that are crystalline in a suitable sense. While the optimal definiton of "crystalline" in this context is open to some discussion (see e.g [BS23, p. 509]), the intended meaning for our purposes is one of the following two variants (relative to a fixed choice of an integer $i \geq 0$):

(a) (abstract) T is a p–torsion subquotient of a G_K–stable lattice in a crystalline $\mathbb{Q}_p$–representation whose Hodge–Tate weights are contained in the interval $[-i, 0]$[1].

(b) (geometric) T is the i–th étale cohomology group with $\mathbb{Z}/p\mathbb{Z}$–coefficients of a proper smooth p–adic formal scheme over $\mathcal{O}_K$ (or a subquotient thereof).

More concretely, we are interested in ramification of such representations. Let G_K^v denote the upper–index higher ramification subgroups of $G_K = \mathrm{Gal}(\overline{K}/K)$. Our main result is the following:

Theorem 1.1. *Assume that K is absolutely unramified. Let T be a mod p crystalline representation in the sense of (a) or (b) above, relative to the integer i. Then G_K^v acts trivially on T when*

$$v > \alpha + \max\left\{ 0,\ \frac{ip}{p^\alpha(p-1)} - \frac{1}{p-1} \right\},$$

where α is the least integer satisfying $p^\alpha > ip/(p-1)$.

Results of this type have a long history, going back to Fontaine's paper [Fon85] on the non–existence of Abelian varietes over $\mathbb{Q}$ with good reduction everywhere. To a large extent, Fontaine's proof is based on a similar type of ramification bound for finite flat group schemes of order p^n (over $\mathcal{O}_K$ for general K). Subsequently, Fontaine [Fon93] and Abrashkin [Abr90] provided another version of ramification bounds for crystalline mod p (mod p^n in [Abr90]) representations in the sense of (a) above, but only

[1] By the results of [EG23], *every* mod p representation of G_K admits a crystalline lift; so without extra restrictions, such as the range of Hodge–Tate weights, the notion of "crystalline" would be meaningless.

when K is absolutely unramified and the bounding integer i satisfies $i < p-1$. The reason for these restrictions is the use of Fontaine–Laffaille theory [FL82], which works well only in this setting.

Of the further developments [BM02, Hat09, Abr15, CL11, Car13], let us explicitly list the extensions to the "abstract semistable" case, that is, the analogue of (a) for semistable representations. Breuil [Bre98] (see also [BM02]) proved such bounds assuming $ie < p-1$ where e is the ramification index of $K/\mathbb{Q}_p$, and under additional assumptions (Griffiths transversality). Hattori's work [Hat09] then removed these extra assumptions and improved the applicable range to $i < p-1$ (with e arbitrary, also in mod p^n version). Finally, Caruso and Liu [CL11] obtained a bound for abstract p^n–torsion semistable representations with e and i arbitrary, using the theory of $(\varphi, \widehat{G})$–modules [Liu10], an enhancement of Breuil–Kisin modules [Kis06] attached to lattices in semistable representations.

It is worth noting that the above results also apply to the geometric setting (b), resp. its semistable analogue, using various comparison theorems [FM87, Car08, LL20]; however, these typically apply only when $ie < p-1$. This was the motivation for the author's previous work [Čou21], where a ramification bound was established for mod p geometric crystalline representations with e and i arbitrary.

While this has been achieved, the obtained result is not optimal: namely, in the setting $ie < p-1$ where the bounds of [Hat09, CL11] apply to étale cohomology of varieties with semistable reduction, the bound of [Čou21] essentially agrees with these semistable bounds. A related question is raised in [CL11] where the authors wonder whether there exists a general ramification bound for subquotients of a crystalline representation. They point out that they do not know any such genuinely crystalline bound beyond the results [Fon93, Abr90] in the Fontaine–Laffaille case.

It is precisely these questions that motivate the present work: while we restrict to the absolutely unramified case ($e = 1$), Theorem 1.1 applies for arbitrarily large i, in both the abstract and the geometric setting, hence goes beyond the scope of Fontaine–Laffaille theory. Moreover, specializing the results of [Hat09, CL11, Čou21] to $e = 1$, the present bound is in fact stronger (see Remark 4.13).

Just like in [Čou21], the key input for the (geometric part of) the proof comes from prismatic cohomology [BMS18, BMS19, BS22]. Let us contrast the two approaches. In [Čou21], to the geometric mod p crystalline representation a pair $(M_{\mathrm{BK}}, M_{\mathrm{inf}})$ was attached, consisting of the mod p versions of Breuil–Kisin and A_{inf}–cohomology (the latter carrying Galois action). The key step in implementing a variant of the strategy [CL11] was then to prove a series of conditions (Cr_s), $s \geq 0$, reflecting the crystalline origin of these modules. These conditions are of the form "$(g-1)M_{\mathrm{BK}} \subseteq I_s M_{\mathrm{inf}}$" for g coming from Galois groups of members of the Kummer tower $\{K(\pi^{1/p^s})\}_s$ associated with the Breuil–Kisin prism $\mathfrak{S}$ and its embedding to the Fontaine prism A_{inf}.

In contrast, the present paper uses the theory of Wach modules [Wac96, Wac97, Col99, Ber04] rather than Breuil–Kisin modules. On the cohomological side, Breuil–Kisin cohomology is replaced by an instance of q–crystalline cohomology [BS22, §16], which we call *Wach cohomology*. Such a replacement is natural: unlike Breul–Kisin modules, Wach modules relate only to crystalline representations.

Roughly speaking, the shift from Breuil–Kisin to Wach modules amounts to replacing the prism $\mathfrak{S}$ by the Wach prism $\mathbb{A} \subseteq A_{\mathrm{inf}}$, which has many consequences. Firstly, the Kummer tower $\{K(\pi^{1/p^s})\}_s$ is replaced by the better–behaved cyclotomic tower $\{K(\mu_{p^s})\}_s$ in our argument. This is what in the end allows us to obtain a stronger ramification bound. Secondly, unlike $\mathfrak{S}$, the subring $\mathbb{A}$ of A_{inf} is stable under Galois action, and ultimately, so are Wach modules. This allows us to replace the use of conditions (Cr_s) by the single condition analogous to (Cr_0), which is in fact part of the definition of a Wach module. On the other hand, the theory of Wach modules works well only for K absolutely unramified, which is why we consider only this case.

The outline of the paper is as follows. In Section 2 we introduce the most relevant background and notation on prisms that we use, as well as notation connected with ramification groups and ramification bounds. The notion of Wach modules, or rather a version of it suited for our purposes, is recalled in Section 3. Here we also define (mod p) Wach cohomology groups and relate them to étale cohomology. Finally, in Section 4, we carry out the proof of Theorem 1.1. We end the paper by an example showing that our bound in general *does not* apply to semistable representations.

Acknowledgement. As will become apparent, the present note is greatly inspired by the work [CL11] of Xavier Caruso and Tong Liu. I am in particular very grateful to Tong Liu for his input through various discussions on this topic, and overall for his encouragement in carrying out this work.

2. Preliminaries

2.1. **Prisms $\mathbb{A}$ and A_{inf}.** We fix a prime p throughout. Let k be a perfect field of characteristic p, and let $K = W(k)[1/p]$ be the associated absolutely unramified p–adic field. Let us denote by $\mathbb{C}_K$ the completed algebraic closure of K, and by $\mathcal{O}_{\mathbb{C}_K}$ its ring of integers. We let G_K denote the absolute Galois goup of K.

For a general discussion of prisms and prismatic cohomology, we refer the reader to [BS22]. Here we only recall that a prism (A, I) is given by a ring A, an invertible ideal $I \subseteq A$ and, assuming A is p–torsion free, a choice of a Frobenius lift $\varphi : A \to A$, subject to certain compatibilities, such as A being (possibly derived) (p, I)–complete.

(1) The central prism of interest is the prism $(\mathbb{A}, I)$ where $\mathbb{A} = W(k)[[q-1]]$ (with q a formal variable, unit in $\mathbb{A}$), and I is the principal ideal generated by
$$F(q) := [p]_q = \frac{q^p - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{p-1}.$$
The Frobenius lift φ on $\mathbb{A}$ is given by the Witt vector Frobenius on $W(k)$ and by $\varphi(q) = q^p$. When $W(k) = \mathbb{Z}_p$, this is the q–crystalline prism from [BS22, Example 1.3 (4)]. To stress the connection with the theory of Wach modules, and to avoid the conflation with the case over $\mathbb{Z}_p$, we refer to $(\mathbb{A}, I)$ as the *Wach prism associated with* $W(k)$.

(2) Another key prism is the Fontaine prism $(A_{\mathrm{inf}}, \operatorname{Ker}\theta)$ (an instance of a perfect prism [BS22, Example 1.3 (2)]). Here $A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}_K^\flat})$ where $\mathcal{O}_{\mathbb{C}_K^\flat}$ is the inverse limit perfection of $\mathcal{O}_{\mathbb{C}_K}/p$. The map $\theta : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_K}$ is the Fontaine's map, determined by sending the Teichmüller lift $[x]$ of any element $x = (x_0 \bmod p, x_0^{1/p} \bmod p, \dots) \in \mathcal{O}_{\mathbb{C}_K^\flat}$ to x_0.

Let us fix a compatible system $(\zeta_{p^s})_s$ of primitive p^s–th roots of unity, which determines the element
$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}^\flat_{\mathbb{C}_K}.$$
There is a map $\mathbb{A} \to A_{\mathrm{inf}}$ given by sending $q - 1$ to $[\varepsilon^{1/p}] - 1$, and it can be shown that $\operatorname{Ker}\theta$ is generated by the image ξ of $[p]_q$ under this map. We thus obtain a map of prisms $\mathbb{A} \to A_{\mathrm{inf}}$; modulo I, this map becomes the inclusion
$$W(k)[[q-1]]/(F(q)) \simeq W(k)[\zeta_p] \to \mathcal{O}_{\mathbb{C}_K}.$$

Lemma 2.1. *The map $\mathbb{A} \to A_{\mathrm{inf}}$ is faithfully flat.*

Proof. Both $\mathbb{A}$ and A_{inf} are p–adically complete with $\mathbb{A}$ Noetherian. Thus, to prove flatness, by [Sta22, Lemma 0912] it is enough to show that $\mathbb{A}/p^n \to A_{\mathrm{inf}}/p^n$ is flat for every n. This latter statement is a special case of [EG23, Proposition 2.2.12]. For faithful flatness, it is enough to observe that the (unique) maximal ideal of A_{inf} lies above the unique maximal ideal $\mathfrak{m}_{\mathbb{A}} = (p, q-1)$. □

For an integer s, denote $K_{p^s} = K(\mu_{p^s})$, and set $K_{p^\infty} = K(\mu_{p^\infty}) = \bigcup_s K_{p^s}$. Denote by Γ the topological group $\mathrm{Gal}(K_{p^\infty}/K)$. Then the cyclotomic character $\chi : \Gamma \to \mathbb{Z}_p^\times$ is an isomorphism, and takes the closed subgroups $\Gamma_s = \mathrm{Gal}(K_{p^\infty}/K_{p^s}) \subseteq \Gamma$ onto $1 + p^s\mathbb{Z}_p$. Γ has natural action on $\mathbb{A}$ via
$$g(q) = q^{\chi(g)}, \ \ g \in \Gamma,$$
making the map $\mathbb{A} \to A_{\mathrm{inf}}$ G_K–equivariant when treating the Γ–action as a G_K–action via the map $G_K \twoheadrightarrow G_K/G_{K_{p^\infty}} \simeq \Gamma$ (the G_K–action on A_{inf} is induced by the one on $\mathcal{O}_{\mathbb{C}_K}$ by functoriality).

We fix an element $\widetilde{\gamma} \in \Gamma$ such that

(1) For every finite s, $\widetilde{\gamma}|_{K_{p^s}}$ generates $\mathrm{Gal}(K_{p^s}/K)$ ($\simeq (\mathbb{Z}/p^s\mathbb{Z})^\times$).
(2) $\gamma = \widetilde{\gamma}^{p-1}$ topologically generates $\mathrm{Gal}(K_{p^\infty}/K_p)$ ($\simeq 1 + p\mathbb{Z}_p$)).

We may even make sure that γ corresponds to $1+p$ via the cyclotomic character, so that we have

$$\gamma(q) = q^{p+1}.$$

For later use, let us also fix the notation G_s to mean the ablosute Galois group of K_{p^s}. Thus, Γ naturally identifies with the quotient G_K/G_∞, and similarly Γ_s corresponds to G_s/G_∞.

2.2. **Ramification groups and Fontaine's property** (P_m)**.** For an algebraic extension F/K, denote by v_F the additive valuation on F normalized by $v_F(F^\times) = \mathbb{Z}$. Given finite extensions $F/E/K$ with F/E Galois, the lower–index numbering on ramification groups of $G = \mathrm{Gal}(E/F)$ we consider is

$$G_{(\lambda)} = \{g \in G \mid v_F(g(x) - x) \geq \lambda\}, \quad \lambda \in \mathbb{R}_{\geq 0}.$$

Note that $G_{(\lambda)} = G_{\lambda-1}$, where G_λ are the usual lower–index ramification groups as in [Ser13, §IV].

For $t \geq 0$, we define the Herbrand function $\varphi_{F/E}(t)$ by

$$\phi_{F/E}(t) = \int_0^t \frac{\mathrm{d}s}{[G_{(1)} : G_{(s)}]},$$

(which makes sense since $G_{(s)} \subseteq G_{(1)}$ for all $s > 0$). This is an increasing, concave, piecewise linear function, and we define $\psi_{F/E}$ to be the inverse function of $\phi_{F/E}$. Then the upper–index ramification subgroups of G are given by

$$G^{(u)} = G_{(\psi_{F/E}(u))}, \quad u \in \mathbb{R}_{\geq 0}.$$

Once again, this numbering is related to the numbering G^u given in [Ser13, § IV] by $G^{(u)} = G^{u-1}$. In particular, the numbering $G^{(u)}$ is compatible with passing to quotients. Given a possibly infinite Galois extension N/E, we may therefore set

$$\mathrm{Gal}(N/E)^{(u)} = \varprojlim_M \mathrm{Gal}(M/E)^{(u)},$$

where M ranges over finite Galois extensions M/E contained in F.

Given an algebraic extension M/K and a real number $m > 0$, we denote by $\mathfrak{a}_M^{>m}$ ($\mathfrak{a}_M^{\geq m}$, resp.) the ideal of all elements $x \in \mathcal{O}_M$ with $v_K(x) > m$ ($v_K(x) \geq m$, resp.). We consider the following condition formulated by Fontaine [Fon85]:

$(P_m^{F/E})$: For any algebraic extension M/E, if there exists an $\mathcal{O}_E$–algebra map $\mathcal{O}_F \to \mathcal{O}_M/\mathfrak{a}_M^{>m}$, then there exists an E–algebra map $F \hookrightarrow M$.

Let us now assume that F/E is finite. We let $\mu_{F/E}$ denote the infimum of all u such that $\mathrm{Gal}(F/E)^{(u)} = \{\mathrm{id}\}$, if any such u exists (typically when F/E is finite). We measure the ramification of F/E in terms of the invariant $\mu_{F/E}$, which is closely connected with the property (P_m):

Proposition 2.2 ([Fon85, Yos10, CL11])**.** *Denote by $e_{F/E}$ the ramification index of F/E, and let $m > 0$ be a real number. If $(\mathrm{P}_m^{F/E})$ holds then $\mu_{F/E} \leq e_{F/E}m$. Moreover, the validity of $(\mathrm{P}_m^{F/E})$ and the value of $\mu_{F/E}$ remain unchanged if E is replaced by any subfield E' of F unramified over E.*

Finally, let us record a lemma on the behavior of μ in towers that will be useful later on.

Lemma 2.3 ([CL11, Lemma 4.3.1])**.** *Let $L/F/E$ be a tower of finite Galois extensions. Then*

$$\mu_{L/E} = \max\left\{\mu_{F/E}, \phi_{F/E}(\mu_{L/F})\right\}.$$

3. Wach cohomology and Wach modules

Recall that given a (bounded) prism (A, I), to a smooth p–adic formal scheme $\mathfrak{X}$ over A/I one can associate the prismatic cohomology $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A)$. Its cohomology groups $\mathrm{H}_{\Delta}(\mathfrak{X}/A)$ are A–modules equipped with, among other structures, a φ_A–semilinear operator φ. When $\mathfrak{X}$ is proper, $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A)$ is represented by a perfect complex (see [BS22, Theorem 1.8]).

In our setting, the relevant variants of prismatic cohomology are the following.

Definition 3.1. Consider a smooth proper p–adic formal scheme $\mathfrak{X}$ over $W(k)$, and denote by $\mathfrak{X}_p$ the base change of $\mathfrak{X}$ to $W(k)[\zeta_p]$. By *Wach cohomology* of $\mathfrak{X}$, we mean the prismatic cohmology $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_p/\mathbb{A})$. The individual Wach cohomology groups are denoted by $\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})$.

The *mod p Wach cohomology of* $\mathfrak{X}$ is given by

$$\overline{\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_p/\mathbb{A})} = \mathsf{R}\Gamma_{\Delta,}(\mathfrak{X}_p/\mathbb{A}) \overset{\mathsf{L}}{\otimes}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}.$$

We denote by $\overline{\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})}$ the individual cohomology groups of the mod p Wach cohomology of $\mathfrak{X}$.

Let us also recall a version of Wach modules suitable for our purposes. From the standard definitions [Wac96, Col99, Ber04], it deviates in that we allow Wach modules that are not necessarily free.

Definition 3.2. A *Wach module of height* $\leq i$ is a finitely generated $\mathbb{A}$–module M endowed with a $\varphi_{\mathbb{A}}$–semilinear map $\varphi : M \to M$ and a continuous, $\mathbb{A}$–semilinear action of Γ compatible with φ and satisfying the following:

(1) The linearization $\varphi_{\mathrm{lin}} = 1 \otimes \varphi : \varphi_{\mathbb{A}}^* M \to M$ of φ admits a map $\psi : M \to \varphi_{\mathbb{A}}^* M$ such that both $\varphi_{\mathrm{lin}} \circ \psi$ and $\psi \circ \varphi_{\mathrm{lin}}$ are given by multiplication by $([q]_p)^i$.
(2) The induced Γ–action on $M/(q-1)M$ is trivial. Equivalently, for every $g \in \Gamma$ we have
$$(g-1)M \subseteq (q-1)M.$$

We are in particular interested in the case when M is annihilated by p, i.e. when M is a module over $\mathbb{A}/p \simeq k[[q-1]]$. We refer to M as *p–torsion Wach module* in this case.

Wach cohomology groups naturally give rise to Wach modules. As stated earlier, φ comes directly from its description as prismatic cohomology. Let us now discuss the Γ–action portion of the data.

Lemma 3.3. *The ideal $I = ([p]_q) \subseteq \mathbb{A}$ is stable under the G_K–action.*

Proof. The map $\mathbb{A} \to \mathbb{A}/I \simeq W(k)[\zeta_p]$ sends q to ζ_p; it is then easy to see that this map is G_K–equivariant. Therefore, the kernel I is necessarily G_K–stable. □

For a smooth proper p–adic formal scheme $\mathfrak{X}$ over $W(k)$, there is a natural G_K–action on $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_p/\mathbb{A})$ given as follows. For $g \in G_K$, acting by g gives a map of prisms $g : (\mathbb{A}, [p]_q) \to (\mathbb{A}, [p]_q)$. By base change of prismatic cohomology [BS22, Thm 1.8 (5)], we obtain an $\mathbb{A}$–linear isomorphism

$$g^* \mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_p/\mathbb{A}) \to \mathsf{R}\Gamma_{\Delta}(g^*\mathfrak{X}_p/\mathbb{A}) = \mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_p/\mathbb{A}) \tag{3.1}$$

where the last identity comes from identifying $g^*\mathfrak{X}_p$ with $\mathfrak{X}_p$ via the canonical isomorphism $g^*\mathfrak{X}_p \to \mathfrak{X}_p$. The above map can then be identified with an $\mathbb{A}$–g–semilinear map

$$g : \mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_p/\mathbb{A}) \to \mathsf{R}\Gamma_{\Delta}(g^*\mathfrak{X}_p/\mathbb{A}),$$

which is the action of g on $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_p/\mathbb{A})$. When g is from $G_{K_{p^\infty}}$, it acts trivially on both $\mathfrak{X}_p$ and $\mathbb{A}$; thus, the above action becomes an action of $G_K/G_{K_{p^\infty}}$, i.e., action of Γ.

The G_K–action on $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}})$ can be described similarly; consequently, it is easy to see that the base-change map

$$\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_p/\mathbb{A}) \widehat{\overset{\mathsf{L}}{\otimes}}_{\mathbb{A}} A_{\mathrm{inf}} \to \mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}})$$

of [BS22, Theorem 1.8 (5)] is G_K–equivariant (of course, this action no longer factors through Γ).

There is a complex $C^\bullet(\mathfrak{X}_p)$ ($C^\bullet(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}})$, resp.) modelling $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_p/\mathbb{A})$ ($\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}})$, resp.) with the following properties:

(1) ($C^\bullet(\mathfrak{X}_p)$ ($C^\bullet(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}})$, resp.) is a perfect complex and consists termwise of flat $\mathbb{A}$-modules ((p,ξ)–completely flat A_{inf}–modules, resp.),
(2) The G_K–action on $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_p/\mathbb{A})$ ($\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}})$, resp.) comes from a ("strict") G_K–action on $C^\bullet(\mathfrak{X}_p)$ ($C^\bullet(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}})$, resp.). In more detail, for every $g \in G_K$ there is an isomorphism $g^*C^\bullet(\mathfrak{X}_p) \xrightarrow{\sim} C^\bullet(g^*\mathfrak{X}_p) = C^\bullet(\mathfrak{X}_p)$ that represents the map (3.1), and which defines a semilinear action of G_K

on $C^\bullet(\mathfrak{X}_p)$ (similarly for $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}})$). Moreover, $G_{K_{p^\infty}}$ acts trivially on $C^\bullet(\mathfrak{X}_p)$, hence we get a Γ–action on $C^\bullet(\mathfrak{X}_p)$.

(3) We have $C^\bullet(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}) \simeq C^\bullet(\mathfrak{X}_p)\widehat{\otimes}_{\mathbb{A}} A_{\mathrm{inf}}$, compatibly with the G_K–action. Here the completed tensor product is computed term–by–term.

The existence of such complexes was established in [Čou21, §2.2] by extending a Čech–Alexander construction of [BS22] from affine case to the case of a general separated formal scheme.

The following proposition is the analogue of the condition (Cr_0) from [Čou21] in our present context.

Proposition 3.4. *For all i and all $g \in \Gamma$, we have*

$$(g-1)C^i(\mathfrak{X}_p) \subseteq (q-1)C^i(\mathfrak{X}_p).$$

Proof. Consider the complex $C^\bullet(\mathfrak{X}_p)/(q-1)C^\bullet(\mathfrak{X}_p)$, where the quotient is computed term-by-term. This is the Čech–Alexander complex computing $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_k/W(k))$, that is, up to $\varphi_{W(k)}$–twist, the crystalline cohomology of the special fiber. The Γ–action on this complex, defined as in (2), on one hand comes from $C^\bullet(\mathfrak{X}_p)$, and on the other is trivial as G_K acts trivially on both $W(k)$ and $\mathfrak{X}_k$. This proves the claim. ☐

As a consequence of (1) above, $\overline{\mathsf{R}\Gamma_{\Delta,}(\mathfrak{X}_p/\mathbb{A})}$ is modelled by the complex $C^\bullet(\mathfrak{X})/pC^\bullet(\mathfrak{X})$ (computed term–by–term). Then we have

Corollary 3.5. *If M is either $\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})$ or $\overline{\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})}$, we have for all $g \in \Gamma$*

$$(g-1)M \subseteq (q-1)M.$$

Consequently, M is a Wach module of height $\leq i$.

Proof. Let $C^\bullet$ be either the complex $C^\bullet(\mathfrak{X})$ or $C^\bullet(\mathfrak{X})/pC^\bullet(\mathfrak{X})$. It suffices to prove the condition $(g-1)Z^i \subseteq (q-1)Z^i$ where Z^i denotes the degree i cocycles in $C^\bullet$. In both cases, $q-1$ is a non–zero divisor on C^j for every j, since $C^j(\mathfrak{X})$ is $\mathbb{A}$–flat and $p, q-1$ is a regular sequence on $\mathbb{A}$.

Given $c \in Z^i$, by Proposition 3.4 we have $(g-1)c = (q-1)c'$ for some $c' \in C^i$, and it is enough to observe that $c' \in Z^i$. This is indeed the case: If ∂ denotes the differential $C^i \to C^{i+1}$, we have

$$(q-1)\partial(c') = \partial((q-1)c') = \partial((g-1)c) = (g-1)\partial(c) = 0,$$

and we may conclude that $\partial(c') = 0$ since $q-1$ is a non–zero divisor on C^{i+1}.

This verifies condition (2) of Definition 3.2, while condition (1) is a general fact about prismatic cohomology [BS22, Theorem 1.8 (6)]. It follows that M is a Wach module of height $\leq i$. ☐

We also need some better control on the action when acting by elements of the subgroup $\Gamma_s \subseteq \Gamma$. In [Čou21], this was done using certain somewhat independent conditions (Cr_s). In the context of Wach modules, we can obtain the control as a consequence of the property (2) of Definition 3.2.

Lemma 3.6. *If M is an $\mathbb{A}$–module with a semilinear Γ–action satisfying $(\gamma-1)M \subseteq (q-1)M$, then the same is true of the module $M' = (q-1)M$.*

Proof. For a given $m \in M$, note that

$$\begin{aligned}(\gamma-1)((q-1)m) &= (\gamma(q)-1)\gamma(m) - (q-1)m \\ &= (\gamma(q)-1)\gamma(m) - (q-1)\gamma(m) + (q-1)\gamma(m) - (q-1)m \\ &= (\gamma(q)-q))\gamma(m) + (q-1)(\gamma(m)-m) \\ &= q(q-1)^p\gamma(m) + (q-1)^2m'\end{aligned}$$

for some $m' \in M$. Thus, we have $(\gamma-1)((q-1)M) \subseteq (q-1)^2M$, as desired. ☐

Proposition 3.7. *For a p–torsion Wach module M, we have*

$$\forall g \in \Gamma_s : (g-1)M \subseteq (q-1)^{p^s}M\ .$$

Proof. When $s = 0$, this is part (2) of Definitian 3.2, so we may assume $s \geq 1$. Since Γ_s is topologically generated by γ^{p^s}, it is enough to show the assertion for γ^{p^s}. Observe that $\gamma^{p^s} - 1 = (\gamma - 1)^{p^s}$ as endomorphisms of M since $pM = 0$. Thus, we need to verify

$$(\gamma - 1)^{p^s} M \subseteq (q-1)^{p^s} M.$$

But this follows by repeated use of Lemma 3.6. □

Finally, let us discuss Galois representations attached to Wach modules in the sense of Definition 3.2.

Definition 3.8. The G_K–module associated with a p–torsion Wach module M is given by

$$T(M) = (M \otimes_{\mathbb{A}} \mathbb{C}_k^\flat)^{\varphi=1},$$

where the map $\mathbb{A} \to \mathcal{O}_{\mathbb{C}_K^\flat} \to \mathbb{C}_K^\flat$ is given by sending q to $\varepsilon^{1/p}$.

In the geometric setting, the representation obtained this way is the appropriate étale cohomology.

Proposition 3.9. *(1) Let T be a $\mathbb{Z}/p\mathbb{Z}[G_K]$-module of the form L/pL for a G_K–stable lattice L in a crystalline representation whose Hodge–Tate weights are in the range $[-i, 0]$. Then $T = T(M)$ for a p–torsion Wach module M.*

(2) For a proper smooth p–adic formal scheme $\mathfrak{X}$ over $W(k)$, we have the following:

(a) $T(\overline{\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})}) = \mathrm{H}^i_{\acute{e}t}(X_{\mathbb{C}_K}, \mathbb{Z}/p\mathbb{Z})$,

(b) $T(\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})/p\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})) = \mathrm{H}^i_{\acute{e}t}(X_{\mathbb{C}_K}, \mathbb{Z}_p)/p\mathrm{H}^i_{\acute{e}t}(X_{\mathbb{C}_K}, \mathbb{Z}_p)$.

Proof. Let us start with (2). The claim (2a) is proved the same way as [FKW21, Lemma 2.1.6] in the setting of Breuil–Kisin cohomology. First, note that $M \mapsto T(M)$ factors as the base chage $M \to M \otimes_{\mathbb{A}} A_{\mathrm{inf}}$ to A_{inf}–cohomology followed by the analogous functor $M_{\mathrm{inf}} \mapsto (M_{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \mathbb{C}_K^\flat)^{\varphi=1}$ of mod p Breuil–Kisin–Fargues modules. Using [BS22, Theorem 1.8], we obtain a long exact sequence

$$\cdots \mathrm{H}^i_{\acute{e}t}(X_{\mathbb{C}_K}, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \overline{\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})} \otimes_{\mathbb{A}} \mathbb{C}_K^\flat \xrightarrow{1-\varphi} \overline{\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})} \otimes_{\mathbb{A}} \mathbb{C}_K^\flat \longrightarrow \mathrm{H}^{i+1}_{\acute{e}t}(X_{\mathbb{C}_K}, \mathbb{Z}/p\mathbb{Z}) \cdots,$$

so it is enough to observe that the map $1 - \varphi$ is surjective (for every i). Since $[p]_q$ becomes invertible in $\mathbb{C}_K^\flat$, $\overline{\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})} \otimes_{\mathbb{A}} \mathbb{C}_K^\flat$ is in fact an étale φ–module, hence of the form $T \otimes \mathbb{C}_K^\flat$ for some finite $\mathbb{Z}/p\mathbb{Z}$–module T, with φ given by the Frobenius on $\mathbb{C}_K^\flat$. It follows that $1 - \varphi$ is surjective.

Repeating a dévisage version of the argument (or simply invoking [Mor16, Theorem 1.1 (vii)] and [BS22, §17])) shows that $T(\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})) = \mathrm{H}^i_{\acute{e}t}(X_{\mathbb{C}_K}, \mathbb{Z}_p)$, where we extend the definition of T to all Wach modules by the formula $T(M) = (M \otimes_{\mathbb{A}} W(\mathbb{C}_K^\flat))^{\varphi=1}$. The second claim now follows from the fact that T takes the right–exact sequence of Wach modules

$$\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A}) \xrightarrow{p} \mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A}) \longrightarrow \mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})/p\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A}) \longrightarrow 0$$

to a right–exact sequence again. This proves (2b).

To prove (1), one uses the result of Berger [Ber04] that the crystalline lattice L is of the form $T(M_0)$ for a Wach module M_0 of height $\leq i$, finite free as an $\mathbb{A}$–module. Proceeding the same way as in the proof of (1), it follows that $T = T(M)$ for the mod p Wach module $M = M_0/pM_0$. □

4. Ramification bound

We now proceed with the proof of the ramification bound. We follow the strategy used in [Čou21] and thus, ultimately the strategy of [CL11], only adapting its use to the case of Wach modules and the tower of extensions $\{K(\mu_{p^s})\}_s$ rather than $\{K(\pi^{1/p^s})\}_s$. Let us fix a p–torsion Wach module of height $\leq i$, denoted by M. In the geometric situation of Section 3, we may take M as $\overline{\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})}$ or as $\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})/p\mathrm{H}^i_{\Delta}(\mathfrak{X}_p/\mathbb{A})$.

Our aim is to provide a bound on $\mu_{L/K}$ where L is the splitting field of $T(M)$, that is, $L - \overline{K}^{\ker \rho}$ where $\rho : G_K \to \mathrm{Aut}(T(M))$ is the Galois representation. Noting that this L does not change, we may replace $T(M)$ by its dual, which is related to M by

$$T(M)^\vee = T^*(M) := \mathrm{Hom}_{\mathbb{A},\varphi}(M, \mathcal{O}_{\mathbb{C}_K^\flat}).$$

Moreover, the functor $M \mapsto T^*(M)$ clearly depends on M only up to $(q-1)$–power–torsion; thus, replacing M by its quotient modulo $(q-1)$–power–torsion, we may assume that M is free as a $k[[q-1]]$–module. From now on, let us denote $T^*(M)$ by T for short.

Let us denote by $v^\flat$ the tilt of the additive valuation v_K on $\mathcal{O}_{\mathbb{C}_K}$, i.e. $v^\flat(x) = v_K(x^\sharp)$ where v_K is the valuation on $\mathcal{O}_{\mathbb{C}_K}$ determined by $v_K(p) = 1$ and where $(-)^\sharp : \mathcal{O}^\flat_{\mathbb{C}_K} \to \mathcal{O}_{\mathbb{C}_K}$ is the multiplicative map $\mathrm{pr}_0 : \mathcal{O}^\flat_{\mathbb{C}_K} = \varprojlim_{x \to x^p} \mathcal{O}_{\mathbb{C}_K} \to \mathcal{O}_{\mathbb{C}_K}$. For a real number $c > 0$, we denote by $\mathfrak{a}^{>c}$ ($\mathfrak{a}^{\geq c}$, resp.) the ideal of all elements $x \in \mathcal{O}^\flat_{\mathbb{C}_K}$ with $v^\flat(x) > c$ ($v^\flat(x) \geq c$, resp.). Clearly every such ideal is stable under the Frobenius map and under the G_K–action.

Definition 4.1. For a real number $c > 0$, let J_c denote the G_K–module

$$J_c = \mathrm{Hom}_{\mathbb{A},\varphi}(M, \mathcal{O}_{\mathbb{C}_K^\flat}/\mathfrak{a}^{>c}).$$

The Galois action on J_s is given by the usual formula

$$g(f)(x) = g(f(g^{-1}x)), \quad g \in G_K, \quad f \in J_c, \quad x \in M.$$

We further set $\rho_c : T \to J_c$ to be the G_K–equivariant map induced by the projection $\mathcal{O}_{\mathbb{C}_K^\flat} \to \mathcal{O}_{\mathbb{C}_K^\flat}/\mathfrak{a}^{>c}$. Similarly, when $c \geq d > 0$, we denote the natural G_K–equivariant map $J_c \to J_d$ by $\rho_{c,d}$, and denote the image of this map by $I_{c,d}$.

From now on, let us fix the numbers

$$b = \frac{i}{p-1}, \quad a = \frac{pi}{p-1} = pb.$$

The next proposition states that J_a, J_b recover T as a G_K–representation completely. It is an analogue of [CL11, Proposition 2.3.3] in our context.

Proposition 4.2. *The map $\rho_b : T \to J_b$ is injective, and the image agrees with $I_{a,b}$.*

Before proceeding to the proof, let us fix auxiliary data for M and notation that will be useful.

Notation 4.3. Let us fix a free basis $e_1, e_2, \dots e_d$ of M. Let $F \in \mathrm{Mat}_{d\times d}(\mathbb{A}/p)$ be the matrix satisfying

$$(\varphi(e_1), \varphi(e_2), \dots, \varphi(e_d)) = (e_1, e_2, \dots, e_d)F.$$

Since M is of height $\leq i$, the submodule of M generated by $\varphi(M)$ contains $([p]_q)^i M = (q-1)^{(p-1)i} M$. Thus, there is a matrix $V \in \mathrm{Mat}_{d\times d}(\mathbb{A}/p)$ such that $FV = (q-1)^{(p-1)i}\mathrm{Id}$.

To simplify formulas, we use underlined notation, e.g. $\underline{x}$, to refer to a length d vector $(x_1, x_2, \dots, x_d)$. Thus, for example, $\underline{e}$ refers to the ordered basis $(e_1, e_2, \dots, e_d)$ of M. If an operation f makes sense for members of an ordered tuple $\underline{x}$, we use $f(\underline{x})$ to refer to the vecor where f is applied term–by–term. Exception to this rule is when $f = v$ is a valuation, in which case $v(\underline{x})$ is a shorthand for $\min_i v(x_i)$.

Proof of Proposition 4.2. Clearly $I_{a,b}$ contains $\mathrm{Im}\,\rho_b$, so the second part of the claim amounts to the converse inclusion. Fix $f : M \to \mathcal{O}_{\mathbb{C}_K^\flat}/\mathfrak{a}^{>a}$ in J_a. We claim that there exists a *unique* $g : M \to \mathcal{O}_{\mathbb{C}_K^\flat}$ in T such that $f \equiv g \pmod{\mathfrak{a}^{>b}}$. This shows both that ρ_b is injective and that its image is $I_{a,b}$.

First, consider an arbitrary lift $\underline{x} \in (\mathcal{O}^\flat_{\mathbb{C}})^{\oplus d}$ of the vector $f(\underline{e})$. By compatibility of f with φ, we have

$$\underline{x}F \equiv \varphi(\underline{x}) \pmod{\mathfrak{a}^{>a}}$$

and, in fact, the congruence holds modulo $\mathfrak{a}^{\geq c}$ for some $c > a$ (that may be chosen arbitrarily close to a). We may therefore write $\varphi(\underline{x}) - \underline{x}F = Q$ for some matrix Q with coefficients in $\mathfrak{a}^{\geq c}$. If there is a unique vector $\underline{y} \in (\mathfrak{a}^{>b})^{\oplus d}$ such that $\underline{x}' = \underline{x} + \underline{y}$ satisfies $\underline{x}'F = \varphi(\underline{x}')$, we are done. From the equation

$$\underline{x}F + \underline{y}F = \underline{x}'F = \varphi(\underline{x}') = \varphi(\underline{x}) + \varphi(\underline{y})$$

we obtain

$$\underline{y}F = \varphi(\underline{x}) - \underline{x}F + \varphi(\underline{y}) = Q + \varphi(\underline{y}),$$

and applying V on the left, we arrive at

$$(\varepsilon - 1)^{\frac{(p-1)i}{p}}\underline{y} = QV + \varphi(\underline{y})V$$

$$\underline{y} = (\varepsilon - 1)^{-\frac{(p-1)i}{p}}QV + (\varepsilon - 1)^{-\frac{(p-1)i}{p}}\varphi(\underline{y})V. \tag{4.2}$$

Using the fact that Q has coefficients in $\mathfrak{a}^{\geq c}$ with $c > a$ (and, thus, $c/p > b$), it is easy to check that the map $C : \underline{y} \mapsto (\varepsilon - 1)^{-\frac{(p-1)i}{p}}QV + (\varepsilon - 1)^{-\frac{(p-1)i}{p}}\varphi(\underline{y})V$ takes $(\mathfrak{a}^{\geq(c/p)})^{\oplus d}$ to $(\mathfrak{a}^{\geq(c/p)})^{\oplus d}$. Moreover, C is a contraction in the sense that if $v^\flat(\underline{y}_1 - \underline{y}_2) \geq t \geq c/p$, then $v^\flat(C(\underline{y}_1) - C(\underline{y}_2)) \geq pt - i \geq t + h(p-1)$ for $h > 0$ such that $c/p \geq i/(p-1) + h$. By Banach contraction principle, there is a unique fixed point $\underline{y} \in \mathfrak{a}^{\geq c/p}$, hence a unique solution to (4.2), and we are done. □

For $c > 0$ and an integer $s \geq 0$, we say that the action on J_c is G_s*–formal* if for all $g \in G_s$, $f \in J_c$ and all $x \in M$, we have $g(f)(x) = g(f(x))$. Equivalently, $f : M \to \mathcal{O}_{\mathbb{C}_K^\flat}/\mathfrak{a}^{>c})$ is invariant for the action of G_s on the source ("formal" here refers to the fact that one may disregard the action of G_s on M and still get the correct action on J_c). The following result is crucial for establishing the bounds.

Proposition 4.4. *The action on J_c is G_s–formal when $p^s > c(p-1)$. In particular, the action on J_b is G_s–formal when $p^s > i$.*

Proof. Let $f \in J_c$ be arbitrary. For every $x \in M$ and $g \in G_s$, we have $g(x) - x = (q-1)^{p^s}y$ for some y. Applying f, we obtain $f(g(x)) - f(x) = (\varepsilon^{1/p} - 1)^{p^s}f(y)$. Since $v^\flat(\varepsilon^{1/p} - 1) = 1/(p-1)$, we infer that $(\varepsilon^{1/p} - 1)^{p^s}f(y) = 0$ in $\mathcal{O}_{\mathbb{C}_K^\flat}/\mathfrak{a}^{>c})$ when $p^s/(p-1) > c$. That is, under the assumption $p^s > c(p-1)$, every $f \in J_c$ is G_s-invariant, as desired. □

We further need a version of Definition 4.1 with restricted coefficients.

Definition 4.5. Fix an integer $s \geq 0$ and a real number c satisfying $p^s > c > 0$. For an algebraic extension $E/K_{p^{s+1}}$, define

$$J_c^{(s)}(E) = \mathrm{Hom}_{\varphi,\mathbb{A}}(M, (\varphi_k^s))^*\mathcal{O}_E/\mathfrak{a}_E^{>c/p^s}).$$

Here φ_k^s denotes the s–th power of the Frobenius of k. Since $1 > c/p^s$, $p = 0$ in $\mathcal{O}_E/\mathfrak{a}_E^{>c/p^s}$ and it is therefore naturally a k-algebra, so the indicated pullback makes sence. We further view $(\varphi_k^s))^*\mathcal{O}_E/\mathfrak{a}_E^{>c/p^s}$ as an $\mathbb{A}/p$–module via $q - 1 \mapsto \zeta_{p^{s+1}} - 1$. When the extension $E/K_{p^{s+1}}$ is Galois, we endow $J_c^{(s),E}$ with the action of G_{s+1}, given by

$$g(f)(x) = g(f(g^{-1}x)), \ \ g \in G_{s+1}, \ f \in J_c^{(s),E}, \ x \in M.$$

Remark 4.6. When $E = \overline{K}$, there is a G_{s+1}–equivariant isomorphism $\mathcal{O}_{\mathbb{C}_K^\flat}/\mathfrak{a}^{>c} \simeq (\varphi_k^s)^*\mathcal{O}_{\overline{K}}/\mathfrak{a}_{\overline{K}}^{>c/p^s}$; consequently, there is an induced isomorphism $J_c \simeq J_c^{(s)}(\overline{K})$ of G_{s+1}–modules. Similarly, when $F/E/K_{p^{s+1}}$ is a tower of algebraic extensions, the map $\mathcal{O}_E/\mathfrak{a}_E^{>c/p^s} \to \mathcal{O}_F/\mathfrak{a}_F^{>c/p^s}$ is injective (note that $\mathfrak{a}_F^{>c/p^s} \cap \mathcal{O}_E = \mathfrak{a}_E^{>c/p^s}$) and it is G_{s+1}–equivariant when both F and E are Galois over $K_{p^{s+1}}$. Thus, we obtain an injection $J_c^{(s)}(E) \to J_c^{(s)}(F)$, which is G_{s+1}–equivariant in the Galois case.

Fixing E and s, for $0 < d \leq c < p^s$ we have the evident map $J_c^{(s)}(E) \to J_d^{(s)}(E)$ induced by the quotient map $\mathcal{O}_{\overline{K}}/\mathfrak{a}_{\overline{K}}^{>c/p^s} \to \mathcal{O}_{\overline{K}}/\mathfrak{a}_{\overline{K}}^{>d/p^s}$. Denote this map by $\rho_{c,d}^{(s)}(E)$, and its image by $I_{c,d}^{(s)}(E)$.

Finally, we introduce a variant where we lift the coefficients to $\mathcal{O}_E$ from $\mathcal{O}_E/\mathfrak{a}_E^{>c/p^s}$. For that purpose, we fix some further notation first.

Notation 4.7. For an integer s with $p^s > i$, let $F_{(s)}, V_{(s)}$ be the images of the matrices F and V, resp., under the map $k[[q-1]] \to (\varphi_k^s)^*\mathcal{O}_{K_{p^{s+1}}}/p$. We then identify $(\varphi_k^s)^*\mathcal{O}_{K_{p^{s+1}}}/p$ with $\mathcal{O}_{K_{p^{s+1}}}/p$ and consider some lifts $\widetilde{F}_{(s)}, \widetilde{V}'_{(s)}$ of $F_{(s)}$ and $V_{(s)}$, resp., to $\mathcal{O}_{K_{p^{s+1}}}$. Then we have

$$\widetilde{F}_{(s)}\widetilde{V}'_{(s)} \equiv (\zeta_{p^{s+1}}-1)^{(p-1)i}\mathrm{Id} \pmod p.$$

It follows that $\widetilde{F}_{(s)}\widetilde{V}'_{(s)} = (\zeta_{p^{s+1}}-1)^{(p-1)i}(\mathrm{Id}+C)$ for a matrix C with entries in $\mathfrak{a}^{>0}_{K_{p^{s+1}}}$ (here we use that $i/p^s < 1 = v_K(p)$). The matrix $\mathrm{Id}+C$ has an inverse of the form $\mathrm{Id}+D$ where D again has entries in $\mathfrak{a}^{>0}_{K_{p^{s+1}}}$ (it is given by $\sum_{n=1}^{\infty}(-C)^n$). Set $\widetilde{V}_{(s)} = \widetilde{V}'_{(s)}(\mathrm{Id}+D)$. The resulting matrices then satisfy the identity (of matrices over $K_{p^{s+1}}$)

$$\widetilde{F}_{(s)}\widetilde{V}_{(s)} = (\zeta_{p^{s+1}}-1)^{(p-1)i}\mathrm{Id}.$$

Definition 4.8. Given an integer s with $p^s > i$ and an algebraic extension $E/K_{p^{s+1}}$, the set $\widetilde{J}^{(s)}(E)$ is defined as

$$\widetilde{J}^{(s)}(E) = \{\underline{\widetilde{x}} = (\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_d) \in \mathcal{O}_E^{\oplus d} \mid \underline{\widetilde{x}^p} = \underline{\widetilde{x}}\widetilde{F}_{(s)}\},$$

where $\underline{\widetilde{x}^p}$ denotes the vector $(\widetilde{x}_1^p, \widetilde{x}_2^p, \dots, \widetilde{x}_d^p)$. When $E/K_{p^{s+1}}$ is Galois, we endow $\widetilde{J}^{(s)}(E)$ with action on entries of $\underline{\widetilde{x}}$.

Given c with $0 < c < p^s$, there is an evident map $\widetilde{\rho}_c^{(s)}(E) : \widetilde{J}^{(s)}(E) \to J_c^{(s)}(E)$. When $E = \overline{K}$, this map is G_{s+1}–equivariant as long as J_c is G_{s+1}–formal. By Proposition 4.4, this is indeed the case under our running assumption $p^s > i$. We need the following enhancement of Proposition 4.2, analogous to [CL11, Lemma 4.1.4].

Proposition 4.9. *Assume $p^s > a$, that is, $(p-1)p^{s-1} > i$. Then for every algebraic extension $E/K_{p^{s+1}}$, the map $\widetilde{\rho}_b^{(s)}(E) : \widetilde{J}^{(s)}(E) \to J_b^{(s)}(E)$ is injective and its image is $I_{a,b}^{(s)}(E)$.*

Proof. We proceed as in the proof of Proposition 4.2. Let us identify all the rings of the form $(\varphi_k^s)^*(\mathcal{O}_E/\mathfrak{a}^{>t})$ appearing in the proof with $\mathcal{O}_E/\mathfrak{a}^{>t}$. Fix $f \in J_b^{(s)}(E)$ and let $\underline{f}(\underline{e}) \in (\mathcal{O}_E/\mathfrak{a}^{>a/p^s})^{\oplus d}$ be the vector of images of the fixed basis $\underline{e}$ from Notation 4.3. Choosing any lift $\underline{x}$ of $\underline{f}(\underline{e})$ to $\mathcal{O}_E^{\oplus d}$, the aim is to show that there is a unique $\underline{y} \in (\mathfrak{a}^{>b/p^s})^{\oplus d}$ such that $\underline{x}+\underline{y} \in \widetilde{J}^{(s)}(E)$.

The required equation then takes the form

$$(\underline{x}+\underline{y})^p = (\underline{x}+\underline{y})\widetilde{F}_{(s)}$$

which, after applying $\widetilde{V}_{(s)}$ on the right and simplifying, becomes

$$\underline{y} = (\zeta_{p^{s+1}}-1)^{-i(p-1)}(\underline{x}+\underline{y})^p\widetilde{V}_{(s)} - \underline{x}$$

(recall that we use the notation for p–th power of vectors component–by–component). Just like in the proof of Proposition 4.2, the goal is to show that the map C given by

$$C(\underline{y}) = (\zeta_{p^{s+1}}-1)^{-i(p-1)}(\underline{x}+\underline{y})^p\widetilde{V}_{(s)} - \underline{x}$$

takes $(\mathfrak{a}_E^{\geq c})^{\oplus d}$ to $(\mathfrak{a}_E^{\geq c})^{\oplus d}$ for c with $c > b$ (and arbitrarily close to b), and that C is a contraction on this space. To choose such c, note that we have $\underline{x}^p \equiv \underline{x}\widetilde{F}_{(s)}$, (mod $\mathfrak{a}_E^{>a/p^s}$), and after applying $\widetilde{V}_{(s)}$ on the right, it follows that $(\zeta_{p^{s+1}}-1)^{-i(p-1)}\underline{x}^p\widetilde{V}_{(s)} \equiv \underline{x}$ (mod $\mathfrak{a}_E^{>t}$) for $t = a/p^s - i/p^s = b/p^s$. Then we may choose c so that this congruence still holds modulo $\mathfrak{a}_E^{\geq c/p^s}$.

The fact that C takes $(\mathfrak{a}_E^{\geq c})^{\oplus d}$ to $(\mathfrak{a}_E^{\geq c})^{\oplus d}$ is then easily seen as follows. Write

$$C(\underline{y}) = \underbrace{(\zeta_{p^{s+1}}-1)^{-i(p-1)}\underline{x}^p\widetilde{V}_{(s)} - \underline{x}}_{\alpha} + \underbrace{(\zeta_{p^{s+1}}-1)^{-i(p-1)}p\underline{y}R\widetilde{V}_{(s)}}_{\beta} + \underbrace{(\zeta_{p^{s+1}}-1)^{-i(p-1)}\underline{y}^p\widetilde{V}_{(s)}}_{\gamma},$$

where $p\underline{y}R$ consists of the mixed terms from the binomial expansion of $(\underline{x}+\underline{y})^p$. Then $\alpha \in (\mathfrak{a}_E^{\geq c})^{\oplus d}$ by the choice of c, and we further have $\beta \in (\mathfrak{a}_E^{\geq t})^{\oplus d}$ for $t \geq 1 + c/p^s - i/p^s$ and $\gamma \in (\mathfrak{a}_E^{\geq u})^{\oplus d}$ for $u \geq cp/p^s - i/p^s$, both of which are bigger than c/p^s.

To show that C is a contraction, let $h_0 = \min\{1, (p-1)c/p^s\}$ and $h = h_0 - i/p^s$ (then $h > 0$). For $\underline{y}_1, \underline{y}_2 \in \mathfrak{a}_E^{\geq c/p^s}$ with $v_K(\underline{y}_1 - \underline{y}_2) = t$, we have

$$\begin{aligned} C(\underline{y}_1) - C(\underline{y}_2) &= (\zeta_{p^{s+1}} - 1)^{-i(p-1)}((\underline{x}+\underline{y}_1)^p - (\underline{x}+\underline{y}_2)^p)\widetilde{V}_{(s)} \\ &= (\zeta_{p^{s+1}} - 1)^{-i(p-1)}(\underbrace{p(\underline{y}_1 - \underline{y}_2)S}_{\delta} + \underbrace{(\underline{y}_1^p - \underline{y}_2^p)}_{\epsilon})\widetilde{V}_{(s)} \end{aligned}$$

The term δ consists of all the mixed terms in binomial expansions of $(\underline{x}+\underline{y}_1)^p$ and $(\underline{x}+\underline{y}_2)^p$ (and it is easy to see that it has the indicated form, with S an integral matrix). The valuation of δ is therefore at least $1+t$, hence $1+t-i/p^s \geq t+h$ after multiplying by $(\zeta_{p^{s+1}}-1)^{-i(p-1)}$. Regarding ϵ, we have

$$\underline{y}_1^p - \underline{y}_2^p = (\underline{y}_1 - \underline{y}_2)(\underline{y}_1^{p-1} + \underline{y}_1^{p-2}\underline{y}_2 + \cdots + \underline{y}_1\underline{y}_2^{p-2} + \underline{y}_2^{p-1})$$

(where operations on vectors are performed component-by-component), and therefore ϵ is of valuation at least $t+(p-1)c/p^s$. After accounting for $(\zeta_{p^{s+1}}-1)^{-i(p-1)}$, this becomes $t+(p-1)c/p^s - i/p^s \geq t+h$. Thus, the valuation of $C(\underline{y}_1) - C(\underline{y}_2)$ is at least $t+h$, showing that C is a contraction on $(\mathfrak{a}_E^{\geq c})^{\oplus d}$ and thus, finishing the proof. □

Let us denote $L_{s+1} = LK_{p^{s+1}} = L[\zeta_{p^{s+1}}]$. Consider s with $p^s > a$, and an algebraic extension $E/K_{p^{s+1}}$. The four canonical maps between $J_a^{(s)}(E), J_b^{(s)}(E), J_a$ and J_b induce the inclusion

$$\iota_{E,s} : I_{a,b}^{(s)}(E) = \rho_{a,b}^{(s)}(E)(J_a^{(s)}(E)) \hookrightarrow \rho_{a,b}(J_a) = I_{a,b}.$$

The next proposition regarding $\iota_{E,s}$ is a basis for establishing validity of Fontaine's property (P_m) in our context; it is a direct analogue of [Čou21, Theorem 5.13] and [CL11, Theorem 4.1.1].

Proposition 4.10. *The map $\iota_{E,s}$ is an isomorphism if and only if $L_{s+1} \subseteq E$.*

Proof. We have a series of G_{s+1}–equivariant bijections

$$\widetilde{J}^{(s)}(\overline{K}) \simeq I_{a,b}^{(s)}(\overline{K}) \simeq I_{a,b} \simeq T,$$

where the indicated isomorphisms use Proposition 4.9, Remark 4.6, and Proposition 4.2, respectively. Similarly, by Proposition 4.9 we have a G_{s+1}–equivariant isomorphism $\widetilde{J}^{(s)}(E) \simeq I_{a,b}^{(s)}(E)$, and we clearly have $\widetilde{J}^{(s)}(E) = \widetilde{J}^{(s)}(\overline{K})^{G_E}$. Thus, the map $\iota_{E,s}$ may be replaced by the inclusion $T^{G_E} \subseteq T$, for which the statement of the proposition is obviously valid. □

Proposition 4.11. *Let s be an integer such that $p^s > a$, and let $m = a/p^s$. Then Fontaine's property $(P_m^{L_{s+1}/K_{s+1}})$ holds.*

Proof. We follow the proof of [Čou21, Proposition 5.14], ultimately based on the arguments of [Hat09, CL11]. By Proposition 2.2, we may replace K_{s+1} by the maximal unramified extension K_{s+1}^{un} inside L_{s+1}, and prove $(P_m^{L_{s+1}/K_{s+1}^{\mathrm{un}}})$ instead.

Let E/K_{s+1}^{un} be an algebraic extension and let $f : \mathcal{O}_{L_{s+1}} \to \mathcal{O}_E/\mathfrak{a}_E^{>m}$ be an $\mathcal{O}_{K_{s+1}^{\mathrm{un}}}$–algebra map. For $c \in \{a, b\}$, we consider the induced map

$$f_c : \mathcal{O}_{L_{s+1}}/\mathfrak{a}_{L_{s+1}}^{>c/p^s} \to \mathcal{O}_E/\mathfrak{a}_E^{>c/p^s}.$$

First, we claim that this map is well–defined an injective. To prove this, consider a uniformizer $\varpi \in L_{s+1}$. The extension $L_{s+1}/K_{s+1}^{\mathrm{un}}$ is totally ramified, so ϖ satisfies an Eisenstein relation of the form (with $e = e(L_{s+1}/K_{s+1})$)

$$\varpi^e = c_1\varpi^{e-1} + c_2\varpi^{e-2} + \cdots + c_{e-1}\varpi + c_e,$$

with $v_K(c_i) \geq 1/(p^s(p-1))$ for all i, and $v_K(c_e) = 1/(p^s(p-1))$. Applying f, the same relation applies to $f = f(\varpi) \in \mathcal{O}_E/\mathfrak{a}_E^{\geq m}$. Choosing a lift $\widetilde{t}$ of t to $\mathcal{O}_E$, we then obtain the relation

$$\widetilde{t}^e = c_1\widetilde{t}^{e-1} + c_2\widetilde{t}^{e-2} + \cdots + c_{e-1}\widetilde{t} + c_e + r,$$

where $r \in \mathfrak{a}_E^{\geq m}$. Since $m \geq 1/(p^s(p-1))$, the valuation of the left–hand side is that of c_e, and it follows that $v_K(\widetilde{t}) = 1/(ep^s(p-1)) = v_K(\varpi)$. We may therofore conclude that

$$\forall N: \quad \varpi^N \in \mathfrak{a}_{L_{s+1}}^{>c/p^s} \text{ if and only if } \frac{N}{ep^s(p-1)} > \frac{c}{p^s} \text{ if and only if } \widetilde{t}^N \in \mathfrak{a}_E^{>c/p^s}.$$

The 'only if' part shows that f_c is well–defined, and the 'if' part shows that it is injective.

Applying $(\varphi_k^s)^*(-)$ to f_a and f_b, one obtains a commutative square

$$\begin{array}{ccc} (\varphi_k^s)^*\mathcal{O}_{L_{s+1}}/\mathfrak{a}_{L_{s+1}}^{>a/p^s} & \xhookrightarrow{(\varphi_k^s)^*(f_a)} & (\varphi_k^s)^*\mathcal{O}_E/\mathfrak{a}_E^{>a/p^s} \\ \downarrow & & \downarrow \\ (\varphi_k^s)^*\mathcal{O}_{L_{s+1}}/\mathfrak{a}_{L_{s+1}}^{>b/p^s} & \xhookrightarrow{(\varphi_k^s)^*(f_b)} & (\varphi_k^s)^*\mathcal{O}_E/\mathfrak{a}_E^{>b/p^s} \end{array},$$

which in turn induces a commutative square

$$\begin{array}{ccc} J_a^{(s)}(L_{s+1}) & \hookrightarrow & J_a^{(s)}(E) \\ \downarrow{\scriptstyle \rho_{a,b}^{(s)}(L_{s+1})} & & \downarrow{\scriptstyle \rho_{a,b}^{(s)}(E)} \\ J_b^{(s)}(L_{s+1}) & \hookrightarrow & J_b^{(s)}(E) \end{array}.$$

Taking images of the vertical maps, we obtain an injection $I_{a,b}^{(s)}(L_{s+1}) \hookrightarrow I_{a,b}^{(s)}(E)$. By Proposition 4.10 applied to L_{s+1}, $I_{a,b}^{(s)}(L_{s+1}) \simeq T$, and therefore the injection $\iota_{E,s}$ from Proposition 4.10 has to be an isomorphism (it is an injection of finite sets where the source has siye at least as much as the target). It follows by Proposition 4.10 again that $L_{s+1} \subseteq E$. This finishes the proof. □

Finally, we are ready to prove the ramification bound in full.

Theorem 4.12. *For*

$$\alpha = \left(\left\lfloor \log_p\left(\frac{ip}{p-1}\right)\right\rfloor + 1\right) \text{ and } \beta = \max\left\{0, \frac{ip}{p^\alpha(p-1)} - \frac{1}{p-1}\right\},$$

one has $\mu_{L/K} \leq 1 + \alpha + \beta$.

Proof. Set $s = \alpha$ so that $p^s > ip/(p-1) = a$. First we estimate $\mu_{L_{s+1}/K}$. By Lemma 2.3, we have

$$\mu_{L_{s+1}/K} = \max\{\mu_{K_{p^{s+1}}/K}, \phi_{K_{p^{s+1}}/K}(\mu_{L_{s+1}/K_{s+1}})\}.$$

Propositions 4.11 and 2.2 show that $\mu_{L_{s+1}/K_{s+1}} \leq p^s(p-1)m = ip$. A classical computation (e.g. [Ser13, §IV]) shows that $\mu_{K_{p^{s+1}}/K} = s+1$, and that $\phi_{K_{p^{s+1}}/K}(t)$ has the last break point given by $\phi_{K_{p^{s+1}}/K}(p^s) = s+1$, with last slope $1/(p^s(p-1))$. Therefore, we may estimate

$$\phi_{K_{p^{s+1}}/K}(t) \leq 1 + s - \frac{1}{p-1} + \frac{t}{p^s(p-1)},$$

and we obtain

$$\mu_{L_{s+1}/K} \leq \max\{1+s, 1+s-\frac{1}{p-1} + \frac{ip}{p^s(p-1)}\} = 1 + s + \max\{0, \frac{ip}{p^s(p-1)} - \frac{1}{p-1}\}.$$

Finally, observing that $\mu_{L/K} \leq \mu_{L_{s+1}/K}$, we obtain the desired bound. □

Remark 4.13. Let us briefly compare the bound from Theorem 4.12 with the results [Hat09, CL11] (i.e., the semistable case) and [Čou21]. For general comparison between the three, see [Čou21, § 5.2]. Here we only summarize that for K absolutely unramified, the bounds of [CL11, Čou21] both become

$$\mu_{L/K} \leq 1 + \alpha + \max\left\{\frac{ip}{p^\alpha(p-1)} - \frac{1}{p^\alpha}, \frac{1}{p-1}\right\}, \tag{4.3}$$

(with the same α as in Theorem 4.12). In order to compare with [Hat09], one needs to further assume $i < p - 1$, in which case all three bounds agree.

On the other hand, Theorem 4.12 gives

$$\mu_{L/K} \leq 1 + \alpha + \max\left\{\frac{ip}{p^\alpha(p-1)} - \frac{1}{p-1},\ 0\right\}, \tag{4.4}$$

which is a stronger bound in all cases (since α is always at least 1).

Example 4.14. To show that Theorem 4.12 in general excludes torsion *semistable* representations, we consider the example from [Hat09] for which the ramificaion bounds of *loc. cit.* are optimal. Let $K = \mathbb{Q}_p$, and consider the Tate curve E_p at p, i.e. the elliptic curve over $\mathbb{Q}_p$ with $E_P(\overline{\mathbb{Q}_p}) = \overline{\mathbb{Q}_p}^\times/p^{\mathbb{Z}}$. It is well–known that E_p has semistable (and not good) reduction.

The set $E_P(\overline{\mathbb{Q}_p})[p]$ can be identified with the set of all p-th roots of unity together with $p^{1/p}$ and all its conjugates. Consequently, the splitting field for $\mathrm{H}^1_{\text{ét}}(E_{p,\mathbb{C}_K}, \mathbb{Z}/p\mathbb{Z})$ is $L = \mathbb{Q}_p(\zeta_p, p^{1/p})$. By [Hat09, Remark 5.5], one has $\mu_{L/K} = 2 + 1/(p-1)$.

On the other hand, the bound from Theorem 4.12 for $i = 1$ would give the stronger estimate $\mu_{L/K} \leq 2$. This shows that the bounds obtained in Theorem 4.12 are "genuinely crystalline", i.e. not satisfied by varieties with semistable reduction in general.

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(Pavel Čoupek) Department of Mathematics, Michigan State University
Email address: coupekpa@msu.edu