Ramification bounds for mod p étale cohomology via prismatic cohomology

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Monodromy and its Applications

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Motivation & Background

Theorem (Fontaine '85)

There is no non–trivial abelian scheme over $\mathbb{Z}$. Equivalently, there is no non–zero abelian variety over $\mathbb{Q}$ with good reduction everywhere.

Key input in proof: let

- $K/\mathbb{Q}_p$ be a finite extension,
- $e = e(K/\mathbb{Q}_p)$ be the absolute ramification index,
- $G_K^{\mu}, \;\; \mu \geq -1$ be the upper–indexed ramification subgroups of $G_K = \mathrm{Gal}(\overline{K}/K)$.

Theorem (Fontaine '85)

Let Γ be a finite flat commutative group scheme over $\mathcal{O}_K$ that is annihilated by p^n. Then G_K^{μ} acts trivially on $\Gamma(\overline{K})$ when

$$\mu > e\left(n + \frac{1}{p-1}\right) - 1.$$

Motivation & Background

Conjecture (Fontaine '85)

Given a smooth proper $\mathcal{O}_K$–scheme X and $T = H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z})$, G_K^{μ} acts trivially on T when

$$\mu > e\left(n + \frac{i}{p-1}\right) - 1.$$

- Special cases proved: – by Fontaine ('93) when $n = e = 1, \quad i < p - 1,$
  – by Abrashkin ('90) when $e = 1, \quad i < p - 1$
- Similar bounds by Hattori ('09), Caruso–Liu ('11):
  – mod p^n reductions of lattices in semistable $\mathbb{Q}_p$–representations
  – applies to $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p\mathbb{Z})$ when X has semistable reduction and $ie < p - 1$

Main result

Theorem (Č.)

Let $\mathcal{X}$ be a smooth and proper formal $\mathcal{O}_K$–scheme, $\mathbb{C}_K = \widehat{\overline{K}}$, $T = H^i_{\acute{e}t}(\mathcal{X}_{\mathbb{C}_K}, \mathbb{Z}/p\mathbb{Z})$. Then G_K^{μ} acts trivially on T when

- $e \leq p$ *and* $\mu > e\left(\left\lfloor \log_p\left(\frac{ip}{p-1}\right)\right\rfloor + 1\right) + e$,
- $e > p$ *and* $\mu > e\left(\left\lfloor \log_p\left(\frac{ie}{p-1}\right)\right\rfloor + 1\right) + p$,
- $i = 1$ *(e, p arbitrary) and* $\mu > e\left(1 + \frac{1}{p-1}\right)$.

Some comparisons:

- $ie < p-1$ or $i = 1$: agrees with Hattori, Caruso–Liu
- $e = 1$ and $i < p-1$: Fontaine, Abrashkin are slightly stronger
- Bound of Caruso ('13) a posteriori applies; which bound is stronger depends on K

Fontaine's condition (P_m)

- Fix L/K finite Galois extension.

$(P_m^{L/K})$

$$\text{For every alg. extension } E/K: \\ \exists\ \mathcal{O}_L \to \mathcal{O}_E/\mathfrak{a}_E^{>m} \ \text{ over } \mathcal{O}_K \ \Rightarrow\ \exists\ L \hookrightarrow E \ \text{ over } K$$

- here $\mathfrak{a}_E^{>m} = \mathcal{O}_E$–elements of K–valuation $> m$

Proposition (Fontaine, Yoshida)

$$\inf\{\mu \mid \mathrm{Gal}(L/K)^{\mu-1} = 1\} = \inf\{m \mid (P_m^{L/K}) \ \textit{holds}\}.$$

"Meta–strategy" of proof:

- To the rep. T, attach associated linear–algebraic data (modules)
- Encode and prove (P_m) in terms of these modules

Breuil–Kisin(–Fargues) modules, prismatic cohomology

- $\mathcal{O}_K \hookrightarrow \mathcal{O}_{\mathbb{C}_K}$ + choice of $\pi \in K$ uniformizer, $\pi_s = \pi^{1/p^s}$, $s \geq 0$ determine

$$\mathfrak{S} = W(k)[[u]] \longrightarrow A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}_K}^\flat)$$
$$u \longmapsto [\underline{\pi}], \quad \underline{\pi} = (\pi_s)_s \in \mathcal{O}_{\mathbb{C}_K}^\flat$$

- A **Breuil–Kisin module** is a fin.-gen. $\mathfrak{S}$–module M_{BK} + a semilinear operator $\varphi : M_{\mathrm{BK}} \to M_{\mathrm{BK}}$ with certain invertibility condition
  - geometrically comes as **Breuil–Kisin cohomology** of Bhatt–Morrow–Scholze:

$$\mathrm{R}\Gamma_{\Delta}(\mathfrak{X}/\mathfrak{S}) \text{ and } \mathrm{R}\Gamma_{\Delta,1}(\mathfrak{X}/\mathfrak{S}) = \mathrm{R}\Gamma_{\Delta}(\mathfrak{X}/\mathfrak{S}) \otimes^{\mathrm{L}} \mathbb{Z}/p\mathbb{Z}$$

- if the scalar extension $M_{\mathrm{inf}} = M_{\mathrm{BK}} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ has G_K–action, we obtain a **Breuil–Kisin–Fargues G_K–module**.
  - geometrically comes as **A_{inf}–cohomology** of Bhatt–Morrow–Scholze:

$$\mathrm{R}\Gamma_{\Delta}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}}) \simeq \mathrm{R}\Gamma_{\Delta}(\mathfrak{X}/\mathfrak{S})\widehat{\otimes}_{\mathfrak{S}} A_{\mathrm{inf}} \text{ and } \mathrm{R}\Gamma_{\Delta,1}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}}) = \mathrm{R}\Gamma_{\Delta}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}}) \otimes^{\mathrm{L}} \mathbb{Z}/p\mathbb{Z}$$

The conditions (Cr_s)

- Set $G_s = \mathrm{Gal}(\overline{K}/K(\pi_s))$, $G_\infty = \bigcap_s G_s$
- Fix $M_{\mathrm{BK}} = H^i_{\Delta,1}(\mathfrak{X}/\mathfrak{S})$, $M_{\mathrm{inf}} = H^i_{\Delta,1}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}})$, and set

$$T^{\mathrm{BK}} = \mathrm{Hom}_{\mathfrak{S},\varphi}(M_{\mathrm{BK}}, \mathcal{O}^\flat_{\mathbb{C}_K}) \simeq T^\vee|_{G_\infty}, \qquad T^{\mathrm{inf}} = \mathrm{Hom}_{A_{\mathrm{inf}},\varphi}(M_{\mathrm{inf}}, \mathcal{O}^\flat_{\mathbb{C}_K}) \simeq T^\vee.$$

- *Idea:* Mutate $T^{\mathrm{BK}}, T^{\mathrm{inf}}$ enough to verify Fontaine's property (P_m)
- *Key step:* relate the G_s–actions on T^{BK} and T^{inf} as follows:

$$\forall g \in G_s : (g-1)M_{\mathrm{BK}} \subseteq \mathfrak{a}^{>c} M_{\mathrm{inf}}$$

- Reminiscent of a crystallinity criterion of Gee–Liu ('19):

(Cr_0) $$\forall g \in G_K : \quad (g-1)M_{\mathrm{BK}} \subseteq ([\underline{\varepsilon}^{1/p}]-1)[\underline{\pi}]M_{\mathrm{inf}}$$

The conditions (Cr_s)

There is a decreasing sequence of ideals $I_s \subseteq A_{\mathrm{inf}},\ s \geq 0$ such that:

Theorem (Č.)

For every i and every $s \geq 0$, one has (termwise on a representing complex)

$$(\mathrm{Cr}_s) \qquad \forall g \in G_s : \quad (g-1)\mathrm{R}\Gamma_{\Delta}(\mathfrak{X}/\mathfrak{S}) \subseteq I_s \mathrm{R}\Gamma_{\Delta}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}}).$$

Consequently,

(1) *For every i, the cohomology groups satisfy*

$$(\mathrm{Cr}_0) \qquad \forall g \in G_K : \quad (g-1)H^i_{\Delta}(\mathfrak{X}/\mathfrak{S}) \subseteq ([\underline{\varepsilon}^{1/p}]-1)[\underline{\pi}]H^i_{\Delta}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}}).$$

(2) *For every i and every $s \geq 0$, one has*

$$\forall g \in G_s : \quad (g-1)H^i_{\Delta,1}(\mathfrak{X}/\mathfrak{S}) \subseteq ([\underline{\varepsilon}^{1/p}]-1)[\underline{\pi}]^{p^s}H^i_{\Delta,1}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}}).$$

Thank you!