MA 16020 Lesson 28: Double integrals II

Recall (integrals over rectangles): Given a function $z = f(x, y)$ of two variables, the double integrals

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\mathrm{d}y, \quad \int_c^d \int_a^b f(x,y)\mathrm{d}y\mathrm{d}x$$

both compute the volume of the region below $z = f(x, y)$ and above the rectangle:

$R =$

Let us denote this common integral by $\iint_R f(x,y)\mathrm{d}A$.

Integrals over regions in a plane. Given a (reasonably complicated/simple) region R in the xy–plane, the integral $\iint_R f(x,y)\mathrm{d}A$ still makes sense. To compute it, we may consider two basic ways of going through all of the points of the region:

Example. Let R be the the interior of the ellipse $4x^2 + y^2 = 4$. We want to compute $\iint_R(\sqrt{4-y^2}+2x)\mathrm{d}A$.

1. **First way:** Give lower and upper limits for the x-coordinates of points present, then give lower and upper limits (posssibly depending on x) for the corresponding y-coordinates.

This way, we obtain

$$\iint_R(\sqrt{4-y^2}+2x)\mathrm{d}A =$$

2. **Second way (the other way round):** Give lower and upper limits for the y-coordinates of points present, then give lower and upper limits (posssibly depending on y) for the corresponding x-coordinates.

This way, we obtain

$$\iint_R (\sqrt{4-y^2} + 2x)\mathrm{d}A =$$

Finally, let us compute the integral:

Exercise 1. Evaluate the integral $\iint_R (x^2 + y^2)\mathrm{d}A$, where R is the region bounded by the lines $y = 2x, x = 5$ and the x-axis.

Exercise 2. Switch the order of integration for the integral

$$\int_0^3 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\mathrm{d}x\mathrm{d}y \ .$$

Exercise 3. Evaluate the integral

$$\iint_R 6\sin^2(x)\mathrm{d}A \ ,$$

where R is the region bouded by the curves $y = \cos(x)$, $x = \pi/6, x = \pi/2$ and the x–axis.

Exercise 4 (time permitting). Evaluate the integral

$$\int_0^2 \int_{x^2}^4 2x\sqrt{3-y^2}\,\mathrm{d}y\mathrm{d}x \ ,$$