Čech complex for crystalline cohomology

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The purpose of this note is to give a proof of the remark [Sta20, 07MM] about Čech–theoretic computation of crystalline cohomology.

Let X be a separated Noetherian scheme over A/I for a dp-$\mathbb{Z}_{(p)}$-algebra (A, I, γ), such that p is nilpotent on X. Recall that the small crystalline site $Cris(X/A)$ is given by:

- Objects: Divided power thickenings (U, T, δ) of X relative to (A, I, γ), that is, the datum of a closed immersion $U \hookrightarrow T$ inducing homeomorphism of underlying topological spaces, such that the associated ideal sheaf $\mathcal{I}$ is endowed with the dp-structure δ, together with a morphism $T \to \operatorname{Spec} A$ of divided power schemes, and an open immersion $U \to X$ over $\operatorname{Spec} A$, that is, a diagram
$$\begin{array}{ccc} U & \hookrightarrow & T \\ \downarrow & & \\ X & & \downarrow \\ \downarrow & & \\ \operatorname{Spec} A/I & \hookrightarrow & \operatorname{Spec} A \end{array}$$
- Morphisms: Divided power morphisms $(U, T, \delta) \to (U', T', \delta')$, making everything commutative.
- topology: Zariski topology, i.e. cover of (U, T, δ) is given by $\{(U_i, T_i, \delta_i) \to (U, T, \delta)\}$, where $T_i \to T$ are open immersions ($U_i \to U$ are open immersions automatically), making everything commutative.

For an affine open $V \subseteq X$, denote by h_V the functor that sends an object $(U, T, \delta) \in Cris(X/A)$ to the set of factorizations of the implicit map $U \hookrightarrow X$ throught $V \subseteq U$, that is,

$$h_V((U, T, \delta)) = \begin{cases} * & \text{if } U \hookrightarrow X \text{ has image contained in } V, \\ \emptyset & \text{otherwise.} \end{cases}$$

When $(U, T, \delta) \to (U', T', \delta')$ is a morphism in $Cris(X/A)$ and $U' \hookrightarrow X$ factors through V, then so does $U \hookrightarrow X$. Thus, the map $h_V((U', T', \delta')) \to h_V((U, T, \delta))$ is well-defined, and h_V is thus a presheaf of sets on $Cris(X/A)$.

Proposition 0.1. *The presheaf h_V is a sheaf.*

Proof. Let $\{(U_i, T_i, \delta_i) \to (U, T, \delta)\}_i$ be a cover. We need to check that the sequence

$$h_V((U, T, \delta)) \to \prod_i h_V((U_i, T_i, \delta_i)) \rightrightarrows \prod_{i,j} h_V((U_i, T_i, \delta_i) \times_{(U,T,\delta)} (U_j, T_j, \delta_j))$$

is an equalizer sequence. Given the fact that the two terms on the right have at most one element, only two things can happen:

(A) $\prod_i h_V((U_i, T_i, \delta_i)) = \emptyset$. Then there is nothing to check, as then $h_V((U, T, \delta)) = \emptyset$ automatically since it maps into the empty set, and the equalizer condition is trivially satisfied.

(B) $\prod_i h_V((U_i, T_i, \delta_i)) \neq \emptyset$. Then the term on the right-hand side is also nonempty. In this case, both are one-element sets, and the maps on the right are thus necessarily the same. Therefore we need to check that $h_V((U, T, \delta))$ is also nonempty, then the sequence above will be an equalizer sequence again trivially. In this case, we know that U_i are subsets of V (via $U_i \hookrightarrow X$) and since they cover U (also viewed as a subset of X via $U \hookrightarrow X$), it follows that U is also a subset of X. Thus, $h_V((U, T, \delta)) \neq \emptyset$, and we are done.

□

Clearly h_X is the final sheaf of $Sh(Cris(X/A))$.

Let $V = \operatorname{Spec} C \subseteq X$ be an affine open subset, and choose a surjective map

$$0 \to J_V \to P_V \to C \to 0$$

where P_V is a free A-algebra. Let (D_V, I_V, δ) be the pd-envelope of (P_V, J_V). For $e \geq 1$ we define $D_{V;e} = D_V / p^e D_V$. Then, for all e sufficiently large, $(D_{V;e}, I_{V;e}, \delta)$ naturally defines an object (V, S_e, δ) of $Cris(X/A)$ (and of $Cris(V/A)$), [Sta20, 07KG].

We define the sheaf $\mathcal{C}_V$ by the formula

$$\mathcal{C}_V((U, T, \delta)) = \varinjlim_e \operatorname{Hom}_{Cris(X/A)}((U, T, \delta), (V, S_e, \delta)).$$

Remark 0.2. 1. The meaning of the definition is that the sheaf $\mathcal{C}_V$ is the sheaf represented by the object given by the dp-ring

$$\widehat{D}_V := \varprojlim_e D_{V;e}, \quad \widehat{I}_V = \operatorname{Im}\Big(\varprojlim_e I_{V;e} \to \widehat{D}_V\Big).$$

The only obstacle is that such a triple $(\widehat{D}_V, \widehat{I}_V, \delta)$ is, strictly speaking, not an object of $Cris(X/A)$ since the map $\widehat{D}_V \to \widehat{D}_V / \widehat{I}_V$ is not a thickening (but rather a pro-thickening) and p is not nilpotent on $\widehat{D}_V$ (only topologically nilpotent). However, ignoring this issue, it is clear that for an affine object (U, T, δ) of $Cris(X/A)$, corresponding to the dp-algebra (B, J, δ) where p is nilpotent, each morphism $(\widehat{D}_V, \widehat{I}_V, \delta) \to (B, J, \delta)$ factors through $(D_{V;e}, I_{V;e}, \delta)$ for some sufficiently large e. Taking $e' \geq e$, these factorizations are compatible, which leads to the formula above.

2. It is clear that $\mathcal{C}_V$ is "extended by $\emptyset$" from a sheaf on $Cris(V/A)$ in the following sense: There is the sheaf $\mathcal{C}'_V$ on $Cris(V/A)$ defined analogously, i.e. by the formula

$$\mathcal{C}'_V((U, T, \delta)) = \varinjlim_e \operatorname{Hom}_{Cris(V/A)}((U, T, \delta), (V, S_e, \delta)).$$

Then we have, for any $(U, T, \delta) \in Cris(X/A)$,

$$\mathcal{C}_V((U, T, \delta)) = \begin{cases} \mathcal{C}'_V((U, T, \delta)) & \text{if } U \subseteq V \text{ (hence } (U, T, \delta) \in Cris(V/A)\text{)}, \\ \emptyset & \text{else.} \end{cases}$$

The symbol $\varinjlim_e$ in the preceding formulas is meant as a filtered colimit operation of presheaves, i.e. computed objectwise. It is therefore important to observe that in the situation at hand, such colimit presheaves are actually sheaves.

Proposition 0.3. *(1) The presheaf $\mathcal{C}_V$ is a sheaf on $Cris(X/A)$.*

(2) $\mathcal{C}_V \to h_V$ is an epimorphism of sheaves.

Proof. Let $\{(U_i, T_i, \delta) \to (U, T, \delta)\}_i$ be a cover in $Cris(X/A)$. As X is assumed Noetherian, U and T are quasi-compact and hence $p^e = 0$ on T for some sufficiently large e. Since $T_i \hookrightarrow T$ are open immersions, $p^e = 0$ on all T_i's and hence on all U_i's. By the same reasoning, the same is true also for all the schemes appearing in the covers $\{(U_{i,i'}, T_{i,i'}, \delta) \to (U_i, T_i, \delta)\}_{i'}$ where $(U_{i,i'}, T_{i,i'}, \delta) := (U_{i'}, T_{i'}, \delta) \times_{(U,T,\delta)} (U_i, T_i, \delta)$. Thus, the sequence for verifying the sheaf axiom

$$\mathcal{C}_V((U, T, \delta)) \to \prod_i \mathcal{C}_V((U_i, T_i, \delta)) \rightrightarrows \prod_{i,i'} \mathcal{C}_V((U_{i,i'}, T_{i,i'}, \delta))$$

is naturally identified with

$$\mathrm{Hom}((U, T, \delta), (V, S_e, \delta)) \to \prod_i \mathrm{Hom}((U_i, T_i, \delta), (V, S_e, \delta)) \rightrightarrows \prod_{i,i'} \mathrm{Hom}((U_{i,i'}, T_{i,i'}, \delta), (V, S_e, \delta)).$$

Since the functor $\mathrm{Hom}(-, (V, S_e, \delta))$ is a sheaf, the latter sequence is an equalizer sequence, hence the former is as well. This proves (1).

Proving (2) amounts to showing that whenever (U, T, δ) has $U \subseteq V$ (so that $h_V((U, T, \delta)) = * \neq \emptyset$), there exists a morphism $(U, T, \delta) \to (V, S_e, \delta)$ in $Cris(X/A)$, for some $e \geq 1$ sufficiently large. We may additionally assume that (U, T, δ) is affine, given by a dp-algebra (B, J, δ). In that case, there is (by definition) a map $P_V \twoheadrightarrow P_V/J_V \simeq C \to B/J$ which admits a lift to $P_V \to B$ since P_V is free. The resulting map of pairs $(P_V, J_V) \to (B, J)$ then induces a morphism of dp-algebras $(D_V, I_V, \delta) \to (B, J, \delta)$ by the universal property of pd-envelopes. Finally, one can take e large enough such that $p^e = 0$ in B. Then the above map factors through a map $(D_{V;e}, I_{V;e}, \delta) \to (B, J, \delta)$, corresponding to the morphism $(U, T, \delta) \to (V, S_e, \delta)$, as desired. □

Now let us choose a (finite) affine open cover $X = \bigcup_j V_j$. For each $n \geq 1$ and each $(j_1, j_2, \ldots, j_n)$, denote $V_{j_1} \cap \cdots \cap V_{j_n} = V_{j_1,\ldots j_n} = \mathrm{Spec}\, C_{j_1,\ldots,j_n}$.

Note that, assuming a good choice of the free algebras P_V in the definition of the complexes $\mathcal{C}_V$, one has

$$\mathcal{C}_{V_{j_1}} \times \mathcal{C}_{V_{j_2}} \times \cdots \times \mathcal{C}_{V_{j_n}} = \mathcal{C}_{V_{j_1,j_2,\ldots,j_n}}.$$

Proposition 0.4. *Given an affine open cover $X = \bigcup_j V_j$, the map $\coprod_j h_{V_j} \to h_X = *$ to the final object is surjective. Consequently, the map $\coprod_j \mathcal{C}_{V_j} \to h_X = *$ is surjective.*

Proof. The "consequently" part follows thanks to Proposition 0.3 (2).

To prove the main statement, it is enough to show that for every object (U, T, δ) there exists a cover $\{(U_i, T_i, \delta_i) \to (U, T, \delta)\}$ such that $\coprod_j^{pre} h_{V_j}((U_i, T_i, \delta_i)) \neq \emptyset$, where $\coprod^{pre}$ denotes the coproduct as presheaves. Identifying U with the corresponding subset of X, the dp-thickenings

$$(U \cap V_j, T \cap V_j, \delta)$$

make sense ($T\cap V_j$ denotes the open subscheme of T whose set of points corresponds, under the homeomorphism $U\to T$, to $U\cap V_j$), and naturally defines an object of $Cris(X/A)$. It is then easy to see that $\{(U\cap V_j, T\cap V_j,\delta)\to(U,T,\delta)\}_j$ is the desired cover, with $h_{V_j}((U\cap V_j, T\cap V_j,\delta))\neq\emptyset$. □

Recall the following notation: Let C be a site and assume that its topology is subcanonical (i.e. all representable presheaves are sheaves). Recall that, when $\mathcal{K}$ is a sheaf of sets on C, the left exact functor (from the category of abelian sheaves on C to the category of abelian groups)

$$\Gamma(\mathcal{K},-):\mathcal{F}\mapsto\Gamma(\mathcal{K},\mathcal{F})=\mathrm{Hom}_{Shv(\mathsf{C})}(\mathcal{K},\mathcal{F})$$

is left exact, has a right derived functor $\mathsf{R}\Gamma(\mathcal{K},-)$, and we denote by $H^i(\mathcal{K},\mathcal{F})$ the i-th cohomology group of $\mathsf{R}\Gamma(\mathcal{K},\mathcal{F})$. When $\mathcal{K}=h_A$ is a represented sheaf, we write $H^i(A,\mathcal{F})=H^i(h_A,\mathcal{F})$. On the other hand, when $\mathcal{K}=*$ is the final sheaf, we have $H^i(\mathcal{K},\mathcal{F})=H^i(\mathsf{C},\mathcal{F})$.

The following vanishing is crucial for our Čech-style computation.

Lemma 0.5. *Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/A}$-modules on $Cris(X/A)$, and let $V\subseteq X$ be an affine open subset. Then $H^q(\mathcal{C}_V,\mathcal{F})=0$ for all $q>0$.*

Proof. We have, functorially in $\mathcal{F}$,

$$\Gamma(\mathcal{C}_V,\mathcal{F})=\mathrm{Hom}_{Shv}(\varinjlim_e h_{(V,S_e,\delta)},\mathcal{F})=\varprojlim_e \mathrm{Hom}_{Shv}(h_{(V,S_e,\delta)},\mathcal{F})=\varprojlim_e \mathcal{F}((V,S_e,\delta)),$$

so that $\Gamma(\mathcal{C}_V,-)$ factors as

$$Shv(Cris(X/A))\xrightarrow{\{\Gamma((V,S_e,\delta),-)\}_e}\mathsf{Ab}^{(\mathbb{N},\geq)}\xrightarrow{\varprojlim}\mathsf{Ab},$$

hence

$$\mathsf{R}\Gamma(\mathcal{C}_V,-)=\big(\mathsf{R}\varprojlim_e\big)\circ\big(\mathsf{R}\{\Gamma((V,S_e,\delta),-)\}_e\big).$$

By [Sta20, 07JJ], we have $H^i((V,S_e,\delta),\mathcal{F})=0$ for all $i>0$ and every $e\geq 1$, that is, for an injective resolution $0\to\mathcal{F}\to\mathcal{E}^\bullet$, the compex $\Gamma((V,S_e,\delta),\mathcal{E}^\bullet)$, which represents $\mathsf{R}\Gamma((V,S_e,\delta),\mathcal{F})$, is acyclic in nonzero degrees. Since $\mathsf{R}\{\Gamma(V,S_e,\delta),-)\}_e(\mathcal{F})$ is represented by the inverse system

$$\Gamma((V,S_1,\delta),\mathcal{E}^\bullet)\leftarrow\Gamma((V,S_2,\delta),\mathcal{E}^\bullet)\leftarrow\Gamma((V,S_3,\delta),\mathcal{E}^\bullet)\leftarrow\dots$$

(interpreted as a chain complex of inverse systems in the obvious manner), the same follows for $\mathsf{R}\{\Gamma(V,S_e,\delta),-)\}_e(\mathcal{F})$, i.e. it is concentrated in degree 0 and hence (quasi-)isomophic to the inverse system $\{\Gamma((V,S_e,\delta),\mathcal{F})\}_e$. Since $\mathcal{F}$ is a crystal in quasi-coherent $\mathcal{O}_{X/A}$-modules, this inverse system has all the transition maps surjective, hence the system is Mittag-Leffler. It follows that $\varprojlim^1_e\Gamma((V,S_e,\delta),\mathcal{F})=0$, that is, $\mathsf{R}\varprojlim_e\{\Gamma((V,S_e,\delta),\mathcal{F})\}_e$ is again concentrated in degree zero and hence (quasi-)isomorphic to $\varprojlim_e\Gamma((V,S_e,\delta),\mathcal{F})=\Gamma(\mathcal{C}_V,\mathcal{F})$.

Thus, we have verified that $\mathsf{R}\Gamma(\mathcal{C}_V,\mathcal{F})$ is concentrated in degree 0, which proves the claim. □

Finally, we put all the pieces above together.

Proposition 0.6. *Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/A}$-modules on $Cris(X/A)$. Then the cohomologies $H^i(Cris(X/A), \mathcal{F})$ can be computed using the Čech complex*

$$(*) \qquad 0 \to \prod_j \Gamma(\mathcal{C}_{V_j}, \mathcal{F}) \to \prod_{j_1, j_2} \Gamma(\mathcal{C}_{V_{j_1, j_2}}, \mathcal{F}) \to \dots$$

Proof. By Lemma 0.4, the epimorphism of sheaves $\coprod_j \mathcal{C}_{V_j} \to *$ is an epimorphism. By [Sta20, 079Z], it follows that there is a spectral sequence with E_1-page

$$E_1^{p,q} = H^q\Big((\coprod_j \mathcal{C}_{V_j})^{\times p}, \mathcal{F}\Big) = H^q\Big(\coprod_{j_1, j_2, \dots, j_p} \mathcal{C}_{V_{j_1, \dots, j_p}}, \mathcal{F}\Big) = \prod_{j_i, \dots, j_p} H^q(\mathcal{C}_{V_{j_1, \dots, j_p}}, \mathcal{F})$$

converging to $H^{p+q}(*, \mathcal{F}) = H^{p+q}(Cris(X/A), \mathcal{F})$.

By Lemma 0.5, $H^q(\mathcal{C}_{V_{j_1, \dots, j_n}}, \mathcal{F}) = 0$ for every $q > 0$ and every n–tuple of indeces $j_1, \dots, j_n$. The first page of the spectral sequence is thus concentrated in a single row of the form (∗), and so the spectral sequence collapses on the second page. This proves the claim. □

References

[Sta20] The Stacks Project Authors, *Stacks project*, available online at `http://stacks.math.columbia.edu`, 2020.

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