Geometric Quadratic Chabauty over number fields

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Plan:

1. Overview of Chabauty–Coleman, Chabauty–Kim program, Quadratic Chabauty
2. Geometric Quadratic Chabauty over $\mathbb{Q}$ (Edixhoven-Lido)
3. Geometric Quadratic Chabauty over number fields (j.w. Lilienfeldt, Xiao, Yao)

Rational points on curves

Let C be a smooth, projective, geometrically connected curve of genus $g \geq 2$ over a number field K.

Theorem (Mordell's conjecture; Faltings '83)

$C(K)$ is a finite set.

Questions of *effectivity* and *explicit methods*:

- How to algorithmically compute $C(K)$?
- How to produce sharp bound?
- How to make optimal bounds in families?
- ...

Chabauty's argument

Let J denote the Jacobian of C. Denote $r = \mathrm{rank}_{\mathbb{Z}} J(K) (< \infty)$ its Mordell-Weil rank.

Theorem (Chabauty '41)

If $r \leq g - 1$ then $\#C(K)$ is finite.

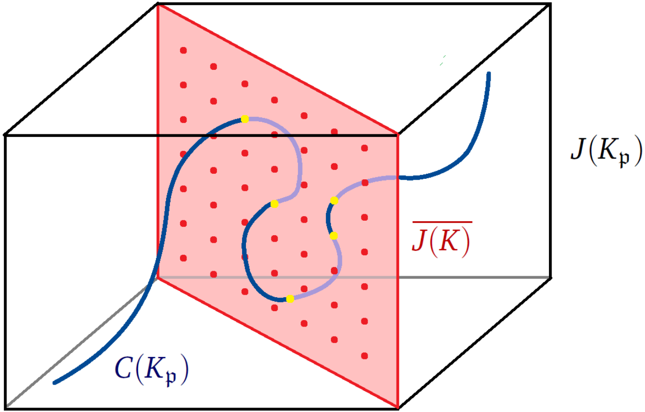
Strategy:
Choose a point $b \in C(K)$, inducing Abel–Jacobi map $j_b : C \hookrightarrow J$, and a prime $\mathfrak{p} \subseteq \mathcal{O}_K$.

$$\begin{array}{ccccc} C(K) & & \hookrightarrow & & C(K_{\mathfrak{p}}) \\ \downarrow j_b & & & \nearrow\!\!\swarrow & \downarrow j_b \\ J(K) & \hookrightarrow & \overline{J(K)} & \longrightarrow & J(K_{\mathfrak{p}}) \end{array}$$

$C(K_{\mathfrak{p}}), \overline{J(K)}$ are $\mathfrak{p}$-adic manifolds of dimensions 1 and $r' \leq r$, resp., in the $\mathfrak{p}$-adic manifold $J(K_{\mathfrak{p}})$ of dimension $g > r'$. Then

$$C(K) \subseteq C(K_{\mathfrak{p}}) \cap \overline{J(K)} \quad \ldots \text{ finite.}$$

Chabauty's argument



Chabauty–Coleman

Theorem (Coleman '85)

Under the same assumption, fix an unramified prime $\mathfrak{p}|p$ of good reduction such that $p > 2g$. Then

$$\#C(K) \leq N(\mathfrak{p}) + 2g(\sqrt{N(\mathfrak{p})} + 1) - 1.$$

Corollary (Coleman '85, McCallum–Poonen 2013)

For $K = \mathbb{Q}$ and a prime p of good reduction with $2g < p$, one further has

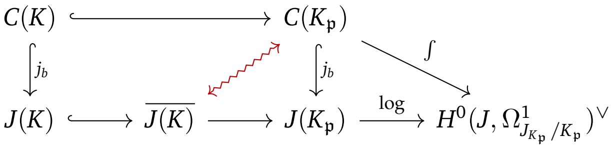
$$\#C(\mathbb{Q}) \leq \#C(\mathbb{F}_p) + (2g - 2).$$

(Some) further improvements:

- Stoll (2006), Katz–Zurieck-Brown (2013): primes of bad reduction
- Katz–Rabinoff–Zurieck-Brown (2016): uniform bound
- ...

Chabauty–Coleman

Strategy:



log, $\int$ are given by $x \mapsto \int_b^x(\bullet)$, the *Coleman integral*. Let

$$V = \{\omega \in H^0(J, \Omega^1_{J_{K_{\mathfrak{p}}}/K_{\mathfrak{p}}}) \mid \int_b^x \omega = 0 \;\; \forall x \in \overline{J(K)}\}.$$

Then

$$C(K_{\mathfrak{p}}) \cap \overline{J(K)} \subseteq \{x \in C(K_{\mathfrak{p}}) \mid \int_b^x j_b^* \omega = 0 \;\; \forall \omega \in V\} =: C(K_{\mathfrak{p}})_1.$$

If $r' < g$, then $V \neq 0$ and a bound on $\#C(K_{\mathfrak{p}})_1$ can be computed.

Chabauty–Coleman

Example (Hirakawa–Matsumura 2019)

Q: *Can a rational right triangle and a rational isosceles triangle have the same area and perimeter?*

Setting up parameters appropriately, this leads to the task of finding $C(\mathbb{Q})$ for

$$C : y^2 = x^6 + 12x^5 - 32x^4 + 52x^2 - 48x + 16 \quad (g = 2)$$

A list of 10 points is

$$\infty^{\pm}, (0, \pm 4), (1, \pm 1), (2, \pm 8), \;\; P^{\pm} = (12/11, \pm 868/11^3).$$

Only P^+ corresponds to a pair of triangles.

Chabauty–Coleman bound ($p = 5$): $\quad \#C(\mathbb{Q}) \leq 10 \Rightarrow$ the list is complete.

The unique pair of triangles has sides $(377, 135, 352)$ and $(366, 366, 132)$, up to scaling.

Restriction-of-Scalars Chabauty

- Siksek (2013)
- Version of Chabauty–Coleman over a number field K

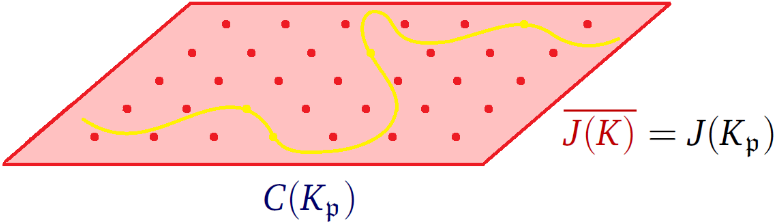
Idea: replace C by $\mathrm{Res}^K_{\mathbb{Q}}(C)$, and work p–adically:

$$\begin{array}{ccccc} C(K)=\mathrm{Res}^K_{\mathbb{Q}}(C)(\mathbb{Q}) & & \hookrightarrow & & \mathrm{Res}^K_{\mathbb{Q}}(C)(\mathbb{Q}_p) \\ \downarrow j_b & & & \leftrightsquigarrow & \downarrow j_b \\ \mathrm{Res}^K_{\mathbb{Q}}(J)(\mathbb{Q}) & \hookrightarrow & \overline{\mathrm{Res}^K_{\mathbb{Q}}(J)(\mathbb{Q})} & \longrightarrow & \mathrm{Res}^K_{\mathbb{Q}}(J)(\mathbb{Q}_p) \end{array}$$

- Generally works when $r \leq (g-1)d, \quad d = [K : \mathbb{Q}]$
- Drawback: it is *not* guaranteed to work.

Chabauty–Coleman

Problem when $r' = g$:



$\rightsquigarrow$ need to "extend the method beyond Jacobian".

Chabauty-Kim program

- Kim (2005, 2009)

Goal: Extend the method beyond the $r < g$ case

$$
\begin{array}{ccccc}
C(K) & \hookrightarrow & C(K_{\mathfrak{p}}) & & \\
\Big\downarrow j_n & & \Big\downarrow j_{n,p} & \searrow^{\text{“}\int\text{”}} & \\
\mathrm{Sel}(U_n) & \xrightarrow{loc_p} & H^1_f(K_{\mathfrak{p}}, U_n) & \xrightarrow{\log_n} & \pi_1^{dR}(C_{K_{\mathfrak{p}}})_n/\mathrm{Fil}^0
\end{array}
$$

U_n = certain unipotent quotients of $\pi_1^{et}(C_{\overline{K}})$

$$C(K_{\mathfrak{p}})_n = j_{n,p}^{-1}(loc_p(\mathrm{Sel}(U_n)))$$

Conjecture (Kim)

For $n >> 0$, $C(K_{\mathfrak{p}})_n$ is finite and coincides with $C(K)$.

Quadratic Chabauty

- Version of $n = 2$ of Kim's program
- uses *double Coleman integrals*: "$z \mapsto \int_b^z \int_b^z (\bullet)$"

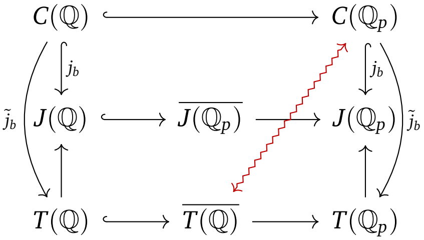
$$
\begin{array}{ccccccc}
C(K) & \hookrightarrow & C(K_{\mathfrak{p}}) & & & \xrightarrow{\int\int} & \\
\downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_{2,p}} & \searrow & & & \\
\mathrm{Sel}(U_2) & \xrightarrow{loc_p} & H^1_f(K_{\mathfrak{p}}, U_2) & \xrightarrow{\log_2} & \pi_1^{dR}(C_{K_{\mathfrak{p}}})_2/\mathrm{Fil}^0 & \longrightarrow & \text{(further quotient)}
\end{array}
$$

- Balakrishnan-Dogra (2016, 2017) - quadratic Chabauty over $\mathbb{Q}$
- Balakrishnan-Dogra-Müller-Tuitman-Vonk (2017)
  - determined rational points of $X_s(13)$, "cursed curve"
- Balakrishnan-Besser-Bianchi-Müller (2019)
  - explicit quadratic Chabauty for hyperelliptic curves over number fields

Geometric quadratic Chabauty over $\mathbb{Q}$

- Edixhoven–Lido (2019)

Goal: Formulate quadratic Chabauty in terms of “simple” geometry:



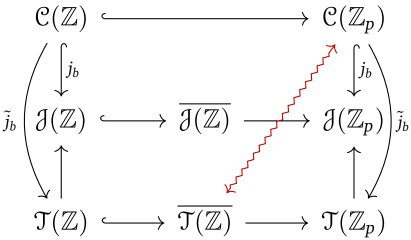
T is a certain $\mathbb{G}_m^{\rho-1}$-torsor on J, $\quad \rho = \operatorname{rank} NS(J)$

Problem: $T(\mathbb{Q})$ has too many points ($\mathbb{Q}^{\times,\rho-1}$ in fibers)

Geometric quadratic Chabauty over $\mathbb{Q}$

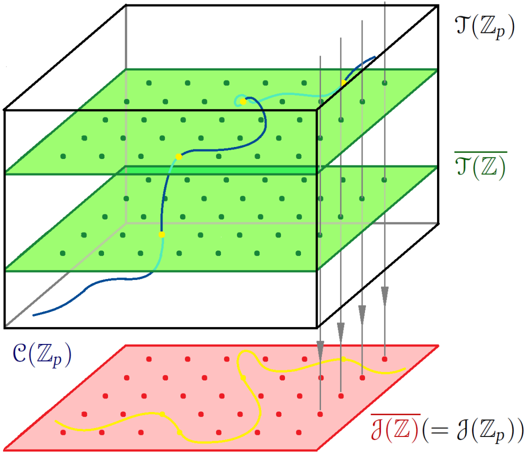
- Edixhoven–Lido (2019)

Goal: Formulate quadratic Chabauty in terms of “simple” geometry



$\mathcal{T}$ is a certain $\mathbb{G}_m^{\rho-1}$-torsor on $\mathcal{J}$,
$\mathcal{J}$ is the Néron model of J,
$\mathcal{C}$ is the smooth locus in a regular proper model of C.

Geometric quadratic Chabauty over $\mathbb{Q}$



Line bundles and $\mathbb{G}_m$–torsors

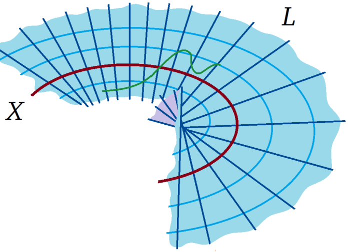
- A $\mathbb{G}_m$–torsor on a scheme X is a scheme T with $\mathbb{G}_m$–action, together with a map $\pi : T \to X$ such that

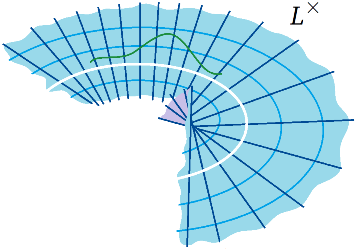
$$\forall\, U \subseteq X \text{ small enough open: } (\pi^{-1}(T) \xrightarrow{\pi} U) \simeq (U \times \mathbb{G}_m \xrightarrow{\mathrm{pr}_U} U)$$

  (+ compatibility conditions).
- *Recall:* There is a 1-1 correspondence between torsors T and line bundles L, given by

$$L \longleftrightarrow T = L^{\times} := L \setminus \text{zero section}$$

- in particular: torsors are parametrized by the Picard scheme $\mathrm{Pic}(X)$





Poincaré biextension

Let $P \to J \times J^\vee$ be the *Poincaré line bundle*:

- $P|_{J\times\{x\}} = L_x$, the line bundle corresponding to $x \in J^\vee(\mathbb{Q})$
- $P|_{J\times\{0\}}, P|_{\{0\}\times J^\vee}$ are trivial line bundles on $J, J^\vee$, resp.
- duality $^\vee$ exchanges J and $J^\vee$ and leaves P unchanged

Then $P^\times$ has the structure of a $\mathbb{G}_m$-*biextension*:

- Given $(x_1, y), (x_2, y) \in J \times J^\vee(S)$, theorem of the cube provides an isomorphism of invertible sheaves, and operation on nowhere vanishing sections

$$(x_1, y)^*\mathcal{P} \otimes_{\mathcal{O}_S} (x_2, y)^*\mathcal{P} \simeq (x_1 + x_2, y)^*\mathcal{P}$$
$$s \otimes t \rightsquigarrow s +_1 t$$

Poincaré biextension

$P^\times$ has the structure of a $\mathbb{G}_m$-*biextension*:

- This defines a group law over $J^\vee$, $+_1 : P^\times \times_{J^\vee} P^\times \to P^\times$ making $P^\times$ an extension

$$0 \to (\mathbb{G}_m)_{J^\vee} \to P^\times \to (J)_{J^\vee} \to 0 \ .$$

- Dually, one has $+_2 : P^\times \times_J P^\times \to P^\times$ and an extension

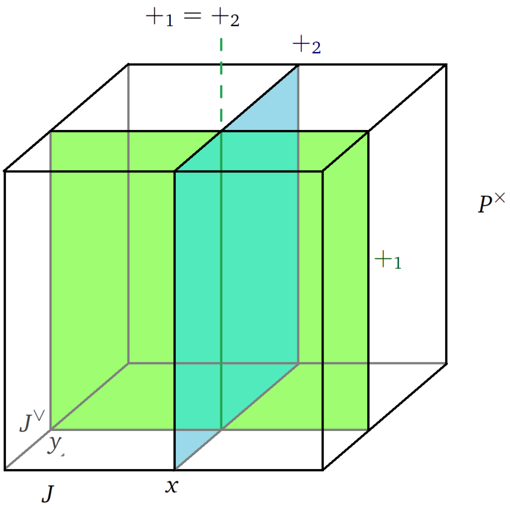
$$0 \to (\mathbb{G}_m)_J \to P^\times \to (J^\vee)_J \to 0 \ .$$

- $+_1, +_2$ are compatible,

$$(a +_1 b) +_2 (c +_1 d) = (a +_2 c) +_1 (b +_2 d)$$

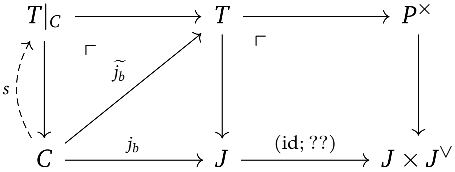
  for $a, b, c, d \in P^\times(S)$ whenever it makes sense.

Poincaré biextension



Constructing T

From now on, assume that $\rho = \operatorname{rank} NS(J) = 2$. We need a non-trivial $\mathbb{G}_m$–torsor T such that C lifts to T – equivalently, such that $T|_C$ is a trivial torsor over C:



Need to find suitable map $?? : J \to J^\vee$ to achieve this.

Constructing T

$$\begin{array}{ccccccccc}
 & & & & \operatorname{Ker} j_b^* & \xrightarrow{\simeq} & \operatorname{Ker} \overline{j_b^*} & & \\
 & & & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & J^\vee & \longrightarrow & \operatorname{Pic}(J) & \longrightarrow & NS(J) & \longrightarrow & 0 \\
 & & \big\downarrow \simeq & & \big\downarrow j_b^* & & \big\downarrow \overline{j_b^*} & & \\
0 & \longrightarrow & J & \longrightarrow & \operatorname{Pic}(C) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Then $\operatorname{rank} \operatorname{Ker} j_b^* = \rho - 1 = 1$, so there is essentialy unique $\mathbb{G}_m$-torsor on J that is trivial over $C \hookrightarrow J$. Moreover, it is of the form

$$T' = (\mathrm{id}_J, \mathrm{t}_c \circ f)^* P^\times, \; f \in \operatorname{Hom}(J, J^\vee)^+, \; c \in J^\vee(\mathbb{Q}),$$

Then $?? = m \cdot \circ \mathrm{t}_c \circ f$ for suitable integer m (in order to spread out over $\mathbb{Z}$)

Parametrization of $\overline{\mathcal{T}(\mathbb{Z})}$

- Work on *residue disks*:

$$\mathcal{X}(\mathbb{Z}_p)_x = \text{ set of all } \widetilde{x} \in \mathcal{X}(\mathbb{Z}_p) \text{ reducing to a given } x \in \mathcal{X}(\mathbb{F}_p),$$
$$\mathcal{X}(\mathbb{Z})_x = \mathcal{X}(\mathbb{Z}_p)_x \cap \mathcal{X}(\mathbb{Z}).$$

$$\begin{array}{ccccccccc}
 & & \mathcal{J}(\mathbb{Z})_0 & & \mathcal{U}(\mathbb{Z})_u & \longrightarrow & \mathcal{U}(\mathbb{Z}_p)_u & \xleftarrow{\simeq} & \mathbb{Z}_p \\
 & & \big\downarrow & \searrow^{\kappa_{\mathbb{Z}}} & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathbb{Z}_p^r & \xrightarrow{\simeq} & \mathcal{J}(\mathbb{Z})_0 \otimes \mathbb{Z}_p & \xrightarrow{\kappa} & \overline{\mathcal{T}(\mathbb{Z})}_{\widetilde{j_b}(u)} & \longrightarrow & \mathcal{T}(\mathbb{Z}_p)_{\widetilde{j_b}(u)} & \xleftarrow{\simeq} & \mathbb{Z}_p^{g+1}
\end{array}$$
$$\overline{\mathcal{T}(\mathbb{Z})}_{\widetilde{j_b}(u)} \leftrightsquigarrow \mathcal{U}(\mathbb{Z}_p)_u$$

- $\kappa_{\mathbb{Z}}$ is constructed using $+_1$ and $+_2$ of $\mathcal{P}^{\times}$
- $\kappa : \mathbb{Z}_p^r \to \mathbb{Z}_p^{g+1}$ can be expressed in terms of p-adically convergent power series.

Parametrization of $\overline{\mathcal{T}(\mathbb{Z})}$

As a consequence, the maps $\mathcal{U}(\mathbb{Z}_p)_u) \xrightarrow{\tilde{j}_b} \mathcal{T}(\mathbb{Z}_p)_{\widetilde{j_b}(u)} \xleftarrow{\kappa} \mathcal{J}(\mathbb{Z})_0 \otimes \mathbb{Z}_p$ induce maps of rings of p-adically convergent power series

$$\mathbb{Z}_p\langle X_1\rangle \xleftarrow[\widetilde{j_b}^*]{} \mathbb{Z}_p\langle X_1, \ldots, X_{g+1}\rangle \xrightarrow{\kappa^*} \mathbb{Z}_p\langle Y_1, \ldots, Y_r\rangle,$$

and upon setting $\quad A = \mathbb{Z}_p\langle Y_1, \ldots, Y_r\rangle / I, \quad I = (\kappa^*(\mathrm{Ker}\, \widetilde{j_b}^*)),$
$\kappa^{-1}\left(\overline{\mathcal{T}(\mathbb{Z}_p)}_{\widetilde{j_b}(u)} \cap \mathcal{U}(\mathbb{Z}_p)_u\right)$ corresponds to $\mathrm{Hom}(A, \mathbb{Z}_p)$.

Theorem (Edixhoven–Lido)

Assuming that $\overline{A} = A \otimes \mathbb{F}_p$ is finite, one has

$$\#\mathcal{U}(\mathbb{Z})_u \leq \dim_{\mathbb{F}_p} \overline{A}.$$

Example (Edixhoven–Lido)

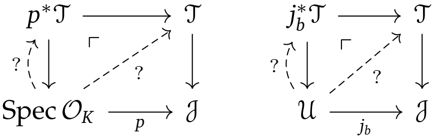
[EL] use the method to explicitly determine $C(\mathbb{Q})$ for a curve C with $g = 2, r = 2, \rho = 2$.
$C = X_0(129)/\langle w_3, w_{43}; \rangle; \quad \#C(\mathbb{Q}) = 14.$

Geometric quadratic Chabauty over number fields

Let $K/\mathbb{Q}$ be a number field, $[K:\mathbb{Q}] = d = r_1 + 2r_2$.

Main obstacles in the number field case:

1. The class group $Cl(K) = \mathrm{Pic}(\mathcal{O}_K)$ may prevent lifting $\mathcal{O}_K$-points and curves:

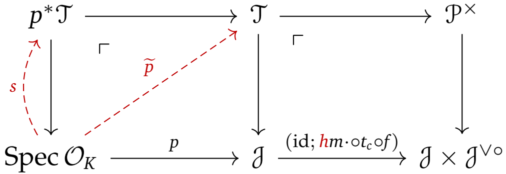


   $\mathrm{Pic}(\mathcal{U}) \to \mathrm{Pic}(C)$ has an h-torsion kernel, $h = \#\mathrm{Pic}(\mathcal{O}_K)$

2. $\mathcal{T}(\mathcal{O}_K) \to \mathcal{J}(\mathcal{O}_K)$ has still too many points, namely $\mathcal{O}_K^{\times,\rho-1} \simeq (\text{torsion}) \times \mathbb{Z}^{\delta(\rho-1)}$, $\delta = r_1 + r_2 - 1$ in (trivial) fibres

Geometric quadratic Chabauty over number fields

Solution to 1 (for $\rho = 2$):



Let $\mathcal{T}' = (\mathrm{id},\, m\cdot \circ t_{c_i} \circ f_i)_i^* \mathcal{P}^\times$.

Then by the biextension law, one can show that

$$\mathcal{T} = (\mathrm{id},\, hm\cdot \circ t_c \circ f)^* \mathcal{P}^\times = (\mathcal{T}')^{\otimes h},$$
$$p^*\mathcal{T} = (p^*\mathcal{T}')^{\otimes h}$$

$\Rightarrow p^*\mathcal{T}$ is an h-th power of a torsor on $\operatorname{Spec} \mathcal{O}_K$, therefore trivial, i.e. s exists.

Geometric quadratic Chabauty over number fields

Solution to 2: We include $\mathcal{O}_K^{\times,\rho-1}$ as part of the parametrization:

$$
\begin{array}{ccc}
\mathcal{J}(\mathcal{O}_K)_0 \times \mathcal{O}_{K,\mathrm{tf}}^{\times,\rho-1} & \xrightarrow{\ \kappa_{\mathbb{Z}}\ } & \mathcal{U}(\mathcal{O}_K)_u \\
\big\downarrow & \searrow & \big\downarrow \\
(\mathcal{J}(\mathcal{O}_K)_0 \times \mathcal{O}_{K,\mathrm{tf}}^{\times,\rho-1}) \otimes \mathbb{Z}_p & \xrightarrow{\ \kappa\ } & \overline{\mathcal{T}(\mathcal{O}_K)_{\widetilde{j_b}(u)}}
\end{array}
$$

Parametrization includes action on fibers by a torsion-free part of $\mathbb{G}_m^{\rho-1}(\mathcal{O}_K)$, $\mathcal{O}_{K,\mathrm{tf}}^{\times,\rho-1} \simeq \mathbb{Z}^{\delta(\rho-1)}$.

Key fact: The $\mathbb{G}_m^{\rho-1}$-action on $\mathcal{P}^{\times,\rho-1}$ is expressible in terms of $+_1, +_2 \Rightarrow \kappa_{\mathbb{Z}}$ is still expressible in terms of $+_1, +_2$, and p-adic interpolation still works.

Summary over number fields

- Fix a rational prime p of good reduction, $e(\mathfrak{p}_i/p) < p-1 \quad \forall \mathfrak{p}_i | p$, and work on "multiresidue disks": fibers of

$$\mathcal{X}(\mathcal{O}_K) \subseteq \mathcal{X}(\prod_i \mathcal{O}_{K,\mathfrak{p}_i}) \to \mathcal{X}(\prod_i \mathbb{F}_{\mathfrak{p}_i})$$

- Parametrization of a "multiresidue" disk now takes the form:

$$\begin{array}{ccccccccc}
 & & \mathcal{J}(\mathcal{O}_K)_0 \times \mathcal{O}_{K,\mathrm{tf}}^{\times,\rho-1} & \overset{\kappa_{\mathbb{Z}}}{\searrow} & \mathcal{U}(\mathcal{O}_K)_u & \longrightarrow & \mathcal{U}(\mathcal{O}_{K,p})_u & \xleftarrow[\simeq]{} & \mathcal{O}_{K,p} \\
 & & \downarrow & & \downarrow & \rightsquigarrow & \downarrow & & \downarrow \\
\mathbb{Z}_p^{r+\delta(\rho-1)} & \xrightarrow{\simeq} & (\mathcal{J}(\mathbb{Z})_0 \times \mathcal{O}_{K,\mathrm{tf}}^{\times,\rho-1}) \otimes \mathbb{Z}_p & \xrightarrow{\kappa} & \overline{\mathcal{T}(\mathcal{O}_K)_{\widetilde{j_b}(u)}} & \to & \mathcal{T}(\mathcal{O}_{K,p})_{\widetilde{j_b}(u)} & \xleftarrow{\simeq} & \mathcal{O}_{K,p}^{g+\rho-1}
\end{array}$$

- $\mathcal{O}_{K,p} = \prod_i \mathcal{O}_{K,\mathfrak{p}_i}$; by a restriction of scalars procedure, or when p splits completely, may view $\mathcal{O}_{K,p} \simeq \mathbb{Z}_p^d$, then κ becomes

$$\kappa : \mathbb{Z}_p^{r+\delta(\rho-1)} \to \mathbb{Z}_p^{d(g+\rho-1)}$$

Main result, Chabauty condition

Theorem (Č., Lilienfeldt, Xiao, Yao 2022)

Given a choice of "multiresidue" disks, there is an explicitely computable $\mathbb{F}_p$-algebra $\overline{A}$ such that, assuming $\overline{A}$ is finite,

$$\#\mathcal{U}(\mathcal{O}_K)_u \leq \dim_{\mathbb{F}_p} \overline{A}.$$

- Method expected to work when $r + \delta(\rho - 1) \leq d(g + \rho - 2)$, equivalently

$$r \leq (g-1)d + (\rho - 1)(r_2 + 1)$$

- agrees with [BBBM] (quadratic Chabauty $/K$, hyperelliptic curves) when $\rho = 2$
- Over $\mathbb{Q}$, this gives $r \leq g + \rho - 2$ - same as [EL], [BD]
- Siksek (linear Chabauty $/K$): condition $r \leq (g-1)d$

Thank you!