MATH 16020 Lesson R: Review of Integration

Spring 2021

Recall. Derivatives tell how much f(x) changes with respect to x.

Two ways to view integration:

(A) Indefinite Integration ("undo" differentiation) also $\int f(x)\,dx = F(x)+C$

Definition. An **antiderivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

Use power rule for derivatives to get the **antiderivative power rule**.

$$\frac{d}{dx}\left[x^{n+1}\right] = (n+1)x^n = \int (n+1)x^n\,dx = x^{n+1} + C$$

C arbitrary, so "absorbs" $1/(n+1)$

$$\xRightarrow{\div(n+1)} \int x^n\,dx = \frac{x^{n+1}}{n+1} + \frac{C}{n+1} = \boxed{\frac{x^{n+1}}{n+1} + C} \Rightarrow \boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1}$$

Other basic antiderivatives come from derivative rules:

Can integrate term-by-term

$$(f(x) \pm g(x))' = f'(x) \pm g'(x) \longrightarrow \int f'(x) \pm g'(x)\,dx = f(x) \pm g(x) + C$$

"b" constant $(b \cdot f(x))' = b \cdot f'(x) \longrightarrow \int b f'(x)\,dx = b\int f'(x)\,dx = bf(x) + C$

$$(\sin(x))' = \cos(x) \longrightarrow \int \cos(x)\,dx = \sin(x) + C$$

$$(\cos(x))' = -\sin(x) \longrightarrow \int \sin(x)\,dx = \cos(x) + C$$

and so on.

Derivatives	Antiderivatives
$\frac{d}{dx}(C) = 0$	$\int 0\, dx = C$
	$\int b\, dx = bx + C$
$\frac{d}{dx}(\ln(x)) = \frac{1}{x},\ x > 0$	$\int \frac{1}{x}\, dx = \ln(\lvert x\rvert) + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x\, dx = e^x + C$
$\frac{d}{dx}(\tan(x)) = \sec^2(x)$	$\int \sec^2(x)\, dx = \tan(x) + C$
$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$	$\int \csc(x)\cot(x)\, dx = -\csc(x) + C$
$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$	$\int \sec(x)\tan(x)\, dx = \sec(x) + C$
$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$	$\int \csc^2(x)\, dx = -\cot(x) + C$

Example 1. Evaluate $\int 6\sec(x)(\sec(x) + 5\tan(x))dx$

$$\int 6\sec^2(x) + 30\sec(x)\tan(x)dx = \int 6\sec^2(x)dx + \int 30\sec(x)\tan(x)dx$$
$$= 6\int \sec^2(x)dx + 30\int \sec(x)\tan(x)dx$$
$$= \boxed{6\tan(x) + 30\sec(x) + C}$$

Example 2. If $y'(x) = \dfrac{x^2 - 1}{\sqrt{x}}$ and $y(4) = 3$, find $y(x)$.

$$y'(x) = \frac{x^2-1}{\sqrt{x}} = \frac{x^2}{\sqrt{x}} - \frac{1}{\sqrt{x}} = x^{3/2} - x^{-1/2}$$
$$\Rightarrow y(x) = \int y'(x)dx = \int x^{3/2} - x^{-1/2}dx = \boxed{\frac{2}{5}x^{5/2} - 2x^{1/2} + C}$$ ← General solution (involves $+C$)

Use $y(4)=3$ to find value of C to get the particular solution.

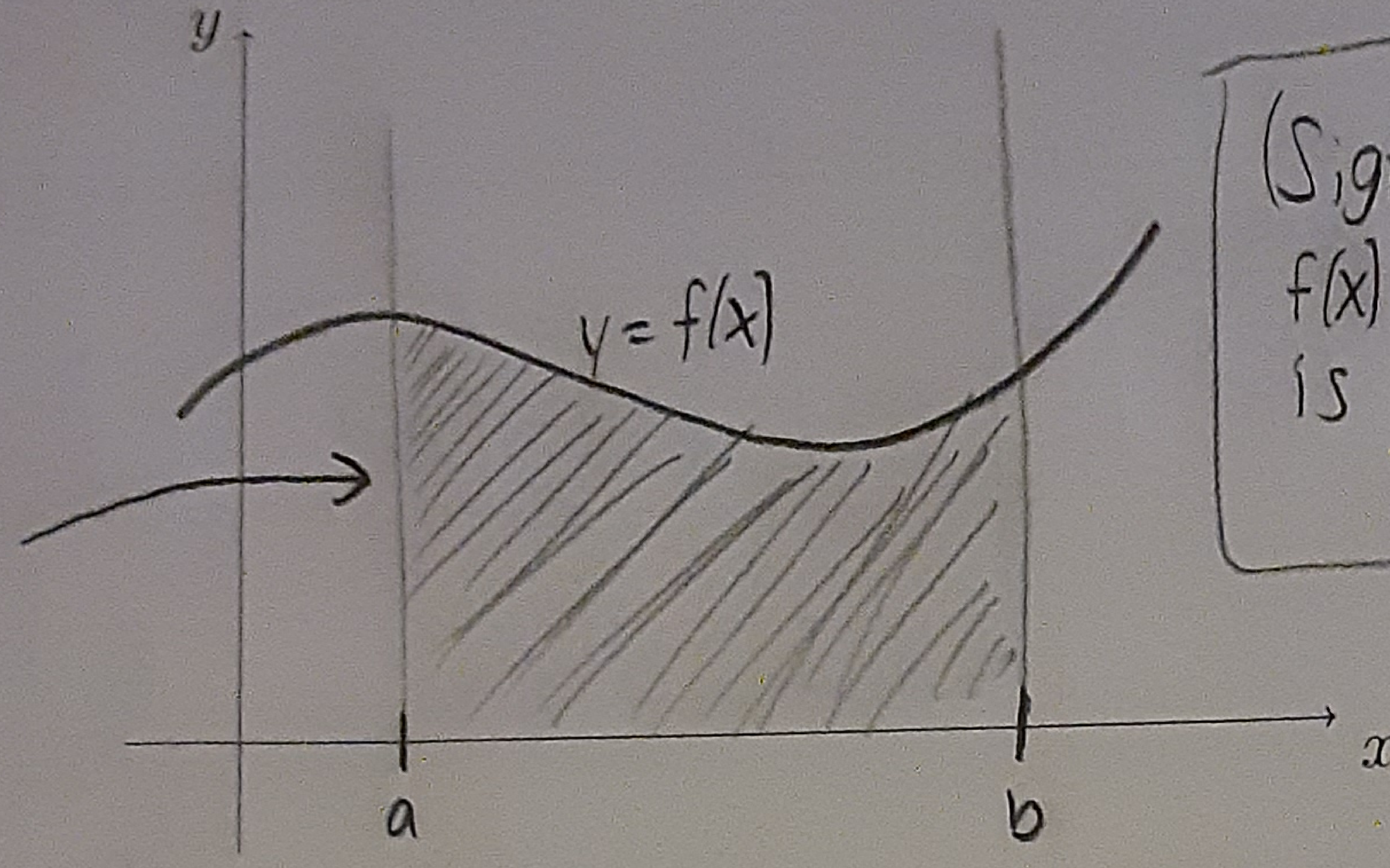
$$y(4) = \frac{2}{5}(4^{5/2}) - 2(4^{1/2}) + C = \frac{2}{5}(32) - 2(2) + C = \frac{64}{5} - 4 + C = 3 \Rightarrow C = 7 - \frac{64}{5} = \frac{-29}{5}$$

$$\Rightarrow \boxed{y(x) = \frac{2}{5}x^{5/2} - 2x^{1/2} - \frac{29}{5}}$$

(B) Definite Integration (Area under curve/measure accumulation)

For these integrals, ALWAYS need an interval or an upper/lower bound

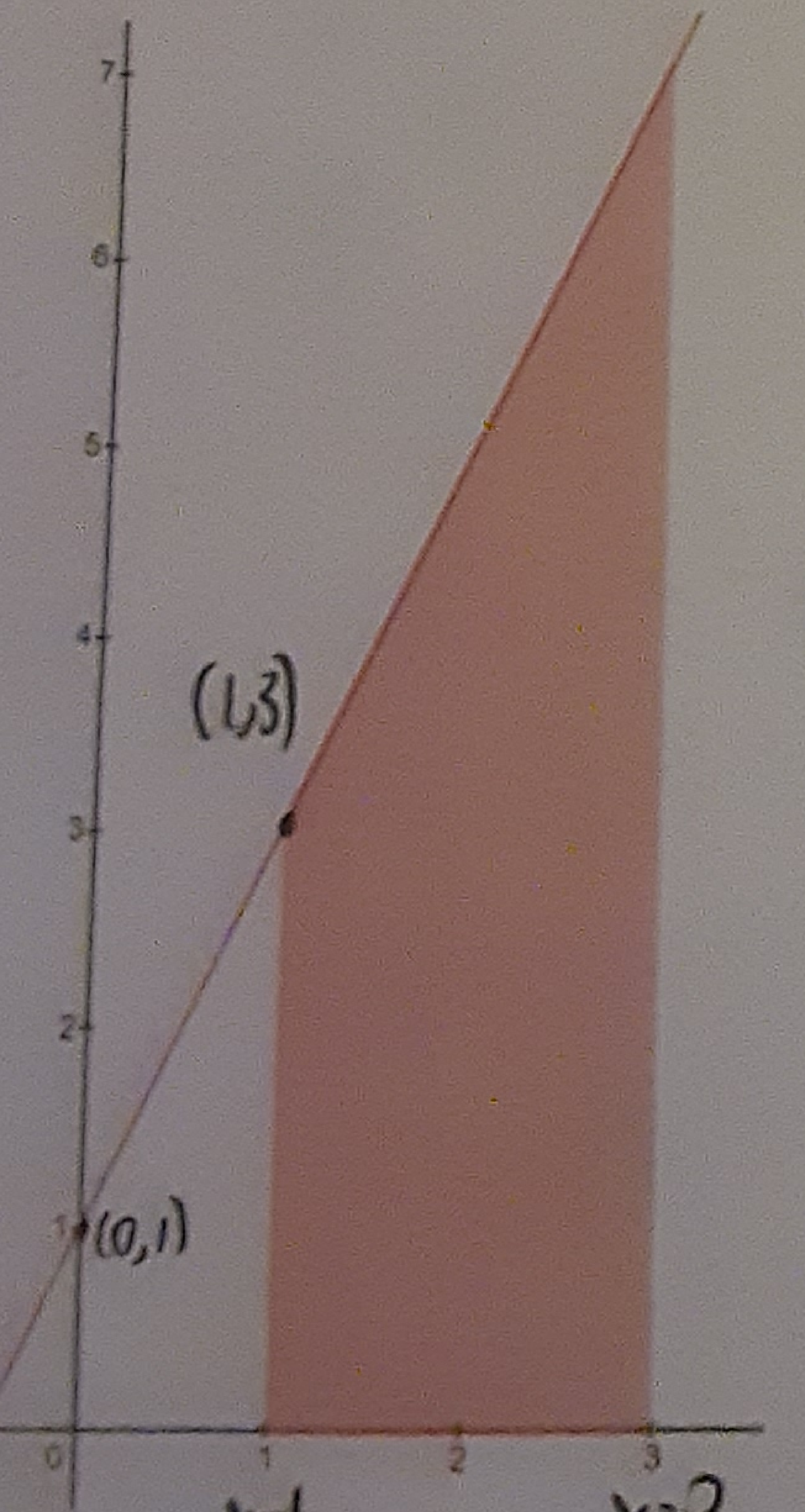
Bounded by $y = f(x)$, $y = 0$, $x = a$, $x = b$



(Signed) area under $f(x)$ from a to b is $\int_a^b f(x)\,dx$.

To evaluate these integrals, need the **Fundamental Theorem of Calculus (FTC)**, stated below: $\boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}$

Example 3. What integral can be used to find the area of the shaded region below? What is this area?



Linear function w/ points $(0,1)$ + $(1,3)$. Find slope 1st.

$m = \frac{3-1}{1-0} = 2 \Rightarrow$

$\Rightarrow$ Slope-intercept form yields $y - 1 = 2x - 0 \Rightarrow \boxed{y = 2x+1}$
(using the point $(0,1)$)

$\Rightarrow$ Integral needed is

$$\int_1^3 2x+1\,dx = \left[x^2+x\right]_1^3 = \left[3^2+3\right] - \left[1^2+1\right] = \boxed{10}$$

Area

Example 4. A strain of bacteria grows at a rate modeled by $r(t) = 8e^t$, where t is in hours since 8AM and $r(t)$ is in number of bacteria per hour.

A. How many bacteria develop from 11AM to 3PM? Round to nearest number of bacteria.

B. How many hours after 8AM will the strain gained 40 more bacteria? Round to nearest hundredth.

(A) 11 AM $\longleftrightarrow t=3$ $\Rightarrow$ Find $\int_3^7 r(t)dt = \int_3^7 8e^t dt = [8e^t]_3^7 = 8e^7 - 8e^3 \approx 8612$
3PM $\longleftrightarrow t=7$

$\boxed{\text{Round ONLY if problem asks to; else, use } 8e^7 - 8e^3.}$

(B) 8AM $\longleftrightarrow t=0$

$\boxed{\text{Let } x = \# \text{ of hrs. after 8AM when 40 more bacteria form.}}$

$\Rightarrow \left(\begin{matrix}\text{Net \# of} \\ \text{new bacteria}\end{matrix}\right) = \int_0^x r(t)dt$

$\Rightarrow 40 = \int_0^x 8e^t dt = 8e^x - 8$

$\Rightarrow 8e^x - 8 = 40$

$\Rightarrow 8e^x = 48$

$\Rightarrow e^x = 6$

$\Rightarrow x = \ln(6) \approx \boxed{1.79 \text{ hrs.}}$