MA 16020 Lesson 9: First-order linear differential equations I

Definition. A **first-order linear differential equation** is a differential equation that can be brought to the form:

$$y' + P(x)y = Q(x)$$

Example 1. The differential equation

$$y' + 5y = e^{2x}$$

is first-order linear. Let's find its general solution:

$$y' + 5y = e^{2x} \quad / \cdot e^{5x}$$ (so far for no apparent reason)

$$e^{5x}y' + \underbrace{5e^{5x}}y = e^{7x}$$

this is $(e^{5x})'$, so the left-hand side is actually $(e^{5x} \cdot y)'$ by the product rule!

$$\rightsquigarrow \quad (e^{5x}y)' = e^{7x},$$ so

$$\underline{e^{5x}y} = \int e^{7x}\,dx = \left| \begin{array}{l} u = 7x \\ du = 7dx \end{array} \right| = \int \frac{e^{u}}{7}\,du = \frac{e^u}{7} + C$$

$$= \underline{\frac{e^{7x}}{7} + C}$$

$$\rightsquigarrow \quad y = \frac{\frac{e^{7x}}{7} + C}{e^{5x}} = e^{-5x}\left(\frac{e^{7x}}{7} + C\right) = \underline{\underline{\frac{e^{2x}}{7} + C \cdot e^{-5x}}},$$

C a general constant.

The employed procedure is called **the method of integrating factors.**

Suppose that we want to solve the differential equation $y' + P(x)y = Q(x)$.

Key step: Find a function $u(x)$ ("integrating factor") such that

$$u(x)\cdot\left(y' + P(x)y\right) = \left(u(x)\cdot y\right)'$$

Such a function can be computed as:

$$\boxed{u(x) = e^{\int P(x)\,dx}}$$

(we can check that this works:

$$u' = \left(e^{\int P(x)dx}\right)' = e^{\int P(x)dx}\cdot\left(\int P(x)\,dx\right)' = \underbrace{e^{\int P(x)dx}}_{u(x)}\cdot P(x) = u(x)P(x)$$

So

$$\left(u(x)y\right)' = u'\cdot y + u\cdot y' = u\cdot P(x)\cdot y + u\cdot y' = u\left(y' + P(x)y\right)$$

)

Method of integrating factors – summary.

1. If necessary, bring the equation to the form $y' + P(x)y = Q(x)$
2. Compute $u(x) = e^{\int P(x)\,dx}$
3. Then proceed solving the equation as follows:

$$y' + P(x)y = Q(x) \quad / \cdot u(x)$$

$$\underbrace{u(x)\left(y' + P(x)y\right)}_{=(u(x)\cdot y)'} = u(x)Q(x)$$

$$\left(u(x)\cdot y\right)' = u(x)Q(x)$$

$$u(x)\cdot y = \int u(x)Q(x)\,dx + C$$

$$\boxed{y = \frac{1}{u(x)}\left(\int u(x)Q(x)\,dx + C\right)}$$

Exercise 2 *(Ex. 4 from last time).* A 800-gallon tank initially contains 600 gallons of pure water. Brine containing 2 pounds of salt per gallon flows into the tank at the rate of 3 gallons per minute, and the well-stirred mixture flows out of the tank at the rate of 2 gallons per minute. What is the amount of salt in the tank after 10 minutes?

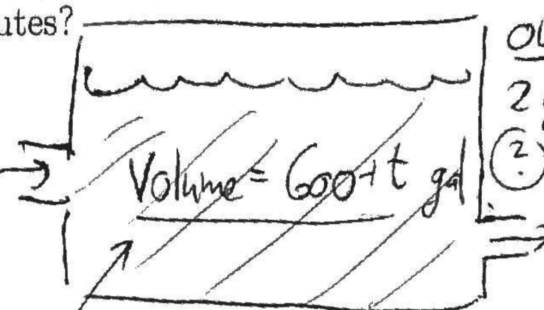
$A(t)$ = quantity of salt after t min

Rate of change = (Rate in) − (Rate out)

Rate in: $(3\ \text{gal/min}) \times (2\ \text{lb/gal}) = 6\ \text{lb/min}$

Rate out $(2\ \text{gal/min}) \times \underbrace{\frac{A(t)}{600+t}}_{\text{concentration in the tank}} = \frac{2A}{600+t}$

In: 3 gal/min, 2 lb/gal



Out: 2 gal/min, (?) lb/gal

(3 gal/min go in, 2 gal/min go out ⇒ Volume increases by 1 gal/min)

$$\leadsto \boxed{\frac{dA}{dt} = 6 - \frac{2A}{600+t}}$$

$$\frac{dA}{dt} + \underbrace{\frac{2}{600+t}}_{P(t)} \cdot A = \underbrace{6}_{Q(t)}$$

$$u(t) = e^{\int \frac{2\,dt}{600+t}} = e^{2\ln(600+t)} = \left(e^{\ln(600+t)}\right)^2 = (600+t)^2$$

$$\leadsto \left((600+t)^2 A\right)' = 6 \cdot (600+t)^2$$

Sub. $u = 600+t$

$$(600+t)^2 A = \int 6(600+t)^2\,dt = 2\cdot(600+t)^3 + C$$

$$\leadsto A = \frac{2\cdot(600+t)^3 + C}{(600+t)^2} = 2\cdot(600+t) + \frac{C}{(600+t)^2}\ 3$$

Determine C -- use $A(0) = 0$:

$$0 = 1200 + \frac{C}{(600)^2}$$

$$\leadsto C = -1200\cdot(600)^2 = 432\,000\,000$$

$$\leadsto A = 2\cdot(600+t) - \frac{432\,000\,000}{(600+t)^2}$$

$$A(10) = 2\cdot(600+10) - \frac{432\,000\,000}{(600+10)^2} =$$

$$\approx 59.02\ \text{lb}$$

Exercise 3. Find the general solution to the diff. equation

$$y' \underbrace{- 4\cot(4x)}_{P(x)} y = \underbrace{5\sin(4x)}_{Q(x)}$$

on the interval $(0, \pi/4)$.

(important: include the "−" sign!)

$$u(x) = e^{\int -4\cot(4x)\,dx}$$

$$\int -4\cot(4x)\,dx = \int -4\,\frac{\cos(4x)}{\sin(4x)}\,dx = \left| \begin{array}{l} u = \sin(4x) \\ du = 4\cos(4x)\,dx \end{array} \right| = \int -\frac{1}{u}\,du =$$

$$= -\ln|u| = -\ln|\sin(4x)| = -\ln(\sin(4x)),$$ since $\sin(4x) > 0$ on $(0, \frac{\pi}{4})$.

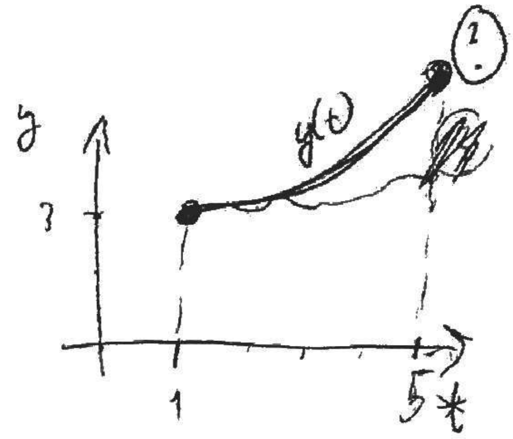
$$\rightsquigarrow \underline{u(x)} = e^{-\ln(\sin 4x)} = \left(e^{\ln(\sin(4x))}\right)^{-1} = (\sin(4x))^{-1} = \underline{\frac{1}{\sin(4x)}}$$

$$\left(\frac{y}{\sin(4x)}\right)' = \frac{5\sin(4x)}{\sin(4x)} = 5$$

$$\rightsquigarrow \frac{y}{\sin(4x)} = \int 5\,dx = 5x + C$$

$$\rightarrow \underline{\underline{y = 5x\sin(4x) + C\sin(4x)}}$$

Exercise 4. Given a function satisfying the equation



$$t^2y' - ty = 3t^2$$

and such that $y(1) = 3$, find $y(5)$.

1) Need „$y' + P(t)\,y = Q(t)$" $\rightarrow$ divide by t^2:

$$\underbrace{y' - \frac{1}{t}}_{P(t)}\cdot y = \underbrace{3}_{Q(t)}$$

we are interested only in $t > 0$

2) $\underline{u(t)} = e^{\int\left(-\frac{1}{t}\right)dt} = e^{-\ln|t|} = e^{-\ln(t)} = \left(e^{\ln(t)}\right)^{-1} = t^{-1} = \underline{\frac{1}{t}}$

3) $$\left(\frac{y}{t}\right)' = \frac{3}{t}$$

$$\frac{y}{t} = \int \frac{3}{t}\,dt = 3\ln(t) + C$$

$\rightarrow$ $\underline{y = 3t\ln(t) + C\cdot t}$ general relation

$\underline{y(1) = 3}$

$$3 = 3\cdot 1\cdot \underbrace{\ln(1)}_{=0} + C\cdot 1 \Rightarrow \boxed{C = 3}$$

$\rightarrow$ $y(t) = 3t\ln(t) + 3t$, and so

$$\underline{y(5) = 15\ln(5) + 15}$$