MA 16020 Lesson 7: Separation of variables II

Recap: A differential equation is called *separable* if it can be brought to the form:

$$f(y)\,\frac{dy}{dx} = g(x)$$

Then we proceed to solve the equation as follows:

$$\left.\begin{aligned} f(y)\,\frac{dy}{dx} &= g(x) \\ \int f(y)\,dy &= \int g(x)\,dx \end{aligned}\right\}\ \text{„separation of variables"}$$

$$F(y) = G(x) + C \quad \left(\begin{array}{l} F \ldots \text{antiderivative of } f, \\ G \ldots \quad - \text{ " } - \quad g \end{array}\right)$$

$\rightsquigarrow$ solve for y.

Exercise 1. Find a general solution to the equation

$$\frac{dy}{dx} = 4x^3(3-y),$$

then find a particular solution satisfying $y(0) = 5$.

$$\frac{dy}{dx} = 4x^3(3-y)$$

$$\frac{1}{3-y}\,\frac{dy}{dx} = 4x^3$$

$$\underbrace{\int \frac{dy}{3-y}}_{?} = \underbrace{\int 4x^3\,dx}_{x^4 + C}$$

$$\int \frac{dy}{3-y} = \left| \begin{array}{l} u = 3-y \\ du = -dy \end{array} \right| = -\int \frac{1}{u}\,du = -\ln|u| = -\ln|3-y|$$

$\rightsquigarrow -\ln|3-y| = x^4 + C$

$\ln|3-y| = -x^4 - C$

$|3-y| = e^{-x^4 - C} = e^{-x^4} \cdot e^{-C}$

$3 - y = \underbrace{\pm e^{-C}}\cdot e^{-x^4}$

new arbitrary constant (call it C again)

$$\underline{y = 3 - C \cdot e^{-x^4}}$$

general solution

$\underline{y(0) = 5}$:

$5 = 3 - C \cdot e^0$
$= 3 - C$

$\rightsquigarrow C = -2$

$$\rightsquigarrow \underline{y = 3 + 2e^{-x^4}}$$

particular solution

Exercise 2. Find a general solution to the equation

$$\frac{dy}{dx} = \frac{2x^3+3}{6y^2}.$$

$$\frac{dy}{dx} = \frac{2x^3+3}{6y^2}$$

$$6y^2 \frac{dy}{dx} = 2x^3+3$$

$$\int 6y^2\,dy = \int (2x^3+3)\,dx$$

$$2y^3 = \frac{x^4}{2} + 3x + C$$

$$y^3 = \frac{x^4}{4} + \frac{3}{2}x + C$$

$$y = \sqrt{\frac{x^4}{4} + \frac{3}{2}x + C}$$

Exercise 3. Find a particular solution to the equation

$$\frac{dy}{dt} = 3e^{3t-2y}$$

such that $y = 0$ when $t = 0$.

$$\frac{dy}{dt} = 3e^{3t-2y} = 3e^{3t}e^{-2y}$$

$$e^{2y}\frac{dy}{dt} = 3e^{3t}$$

$$\int e^{2y}\,dy = \int 3e^{3t}\,dt$$

$$\frac{e^{2y}}{2} = e^{3t} + C$$

$$e^{2y} = 2e^{3t} + C \quad (\leftarrow \text{„new" } C)$$

$$2y = \ln(2e^{3t}+C)$$

$$y = \frac{1}{2}\ln(2e^{3t}+C)$$

$y(0)=0$:

$$0 = \frac{1}{2}\ln(2\cdot 1 + C)$$

$$0 = \ln(2+C) \Rightarrow 2+C = 1$$

$C=-1$, and

$$y = \frac{1}{2}\ln(2e^{3t}-1)$$

Exercise 4. A wet sweater drying in the sun loses its moisture at a rate proportional to its moisture content. After 1 hour, the sweater lost 32% of its original moisture content. How long will it take for the sweater to lose 75% of its original moisture content?

$M(t)$ = amount of moisture after t hours

Know:
- $M(0) = 1$
- $M(1) = 1 - 0.32 = 0.68$
- $\boxed{\frac{dM}{dt} = -kM}$, k constant of proportionality

Want: t such that $M(t) = 0.25$

$$\frac{dM}{dt} = -kM$$

$$\int \frac{dM}{M} = \int -k\,dt$$

$$\ln|M| = -kt + C$$

$$M = \underbrace{\pm e^C}_{"C"} \cdot \underbrace{(e^{-k})}_{"D"}{}^t = C \cdot D^t$$

Determine C: $M(0) = 1$, so

$$1 = C \cdot D^0 = C \cdot 1 = C$$

Determine D: $M(1) = 0.68$, so

$$0.68 = \underbrace{C}_{=1} \cdot D^1 = D$$

$$\rightsquigarrow M(t) = (0.68)^t$$

Finally: Want t s.t. $M(t) = 0.25$

$$(0.68)^t = 0.25 \quad / \ln(-)$$

$$t \cdot \ln(0.68) = \ln(0.25)$$

$$t = \frac{\ln(0.25)}{\ln(0.68)} \approx 3.595 \text{ hours}$$

Exercise 5. A newly created ceramic pot has an inital temperature $1547°F$. Upon placing it into a room with constant temperature $72°F$, after one hour the temperature of the pot is $922°F$. What is the temperature of the pot after 5 hours?
(Recall *Newton's law of cooling:* $\frac{dT}{dt} = -k(T - \underbrace{S}_{\text{temp. of surroundings}})$)

$T(t)$ = temperature of the pot after t hours

<u>Know:</u> $T(0) = 1547°F$
$T(1) = 922°F$
$S = 72°F$

<u>Want:</u> $T(5)$

$$\frac{dT}{dt} = -k(T-72)$$

$$\int \frac{dT}{T-72} = \int (-k)\,dt$$

$$\ln|T-72| = -kt + C$$

$$T - 72 = \underbrace{\pm e^C}_{C} \cdot \underbrace{(e^{-k})}_{D}{}^t = C \cdot D^t$$

$$\Rightarrow T = 72 + C \cdot D^t$$

$T(0) = 1547°F$:

$$1547 = 72 + C \cdot D^0 = 72 + C$$

$$\Rightarrow C = 1547 - 72 = 1475$$

$T(1) = 922°F$:

$$922 = 72 + 1475 \cdot D^1$$

$$\Rightarrow D = \frac{922-72}{1475} = \frac{850}{1475} = \frac{34}{59} \quad (\approx 0.576)$$

$$T(5) = 72 + 1475 \cdot \left(\frac{34}{59}\right)^5 \approx$$

$$\approx 165.740$$