MATH 16020 Lesson 6: Separation of Variables I

Spring 2021

Definition. A **differential equation (DE)** is an equation relating a function to one (or more) of its derivatives.

Examples: ✓ $\frac{dy}{dx} = 8y$, ✓ $y' = t\cos(y)$, X $y' = x^4y + xy^2$, X $\frac{dy}{dt} = \sin(t)y + t^3$

Definition. A DE is **separable** if it has the form $y' = f(x)g(y)$ (1st two are, last 2 not)

Example. Show $\frac{dy}{dx} = x^3e^{y-x^4}$ is separable.
$= x^3e^ye^{-x^4} \Rightarrow \frac{dy}{dx} = \underbrace{x^3e^{-x^4}}_{f(x)}\underbrace{e^y}_{g(y)}$

Definition. A **solution** to a DE is a function that can be plugged into a DE + have it be true.

A **particular solution** to a DE is a sol'n with no arbitrary constant.

Example. A particular solution to $\frac{dy}{dx} = 3y$ is $y(x) = 2e^{3x}$ as shown below:

$y'(x) = 6e^{3x}$, $3y(x) = 6e^{3x} \Rightarrow y' = 3y$

Example 1. Solve $y' = ky$ if $y(0) = 6$ and $y'(0) = 12$.

Let $y = y(x) \Rightarrow y'(x) = ky(x)$ for every x, so:
$y'(0) = ky(0) \Rightarrow 12 = k \cdot 6 \Rightarrow \boxed{k=2} \Rightarrow \boxed{\frac{dy}{dx} = 2y}$ ← Separable so divide y to other side

$\frac{1}{y}\frac{dy}{dx} = 2 \Rightarrow \frac{1}{y}dy = 2dx \Rightarrow \int \frac{1}{y}dy = \int 2dx \Rightarrow \ln|y| = 2x + C \Rightarrow |y| = e^{2x+C} = e^{2x}e^{C}$

"$\times dx$"

$\Rightarrow |y| = Ce^{2x}$, "C" arbitrary

$\downarrow y(0) = 6$

$6 = Ce^0 = C \Rightarrow C = 6$

$\Rightarrow |y(x)| = 6e^{2x}$

This technique is called separation of variables.
(put $g(y)$, dy on one side, $f(x)$, dx on other side)

$\Rightarrow |y(x)| = 6e^{2x} \Rightarrow y(x) = 6e^{2x}$ or $y(x) = -6e^{2x}$

$\Rightarrow \boxed{y(x) = 6e^{2x}}$ because $y(0) = 6 > 0$.

$\boxed{\text{IMPORTANT: General sol'n to } y' = ky \text{ is } y(x) = Ce^{kx}, \text{ where } C = y(0)}$

Example 2. Solve the differential equation below where $y = 2$ if $t = 1$.

$$\frac{dy}{dt} = \frac{\ln(t)}{3y}$$

$\frac{dy}{dt} = \frac{\ln(t)}{3y} \Rightarrow y\,dy = \frac{\ln(t)}{3}dt \Rightarrow \int y\,dy = \frac{1}{3}\int \ln(t)dt = \frac{1}{3}\left[t\ln(t) - \int dt\right]$

$= \frac{1}{3}t\ln(t) - \frac{1}{3}t + \frac{C}{3}$

$\boxed{\text{Only option is int. by parts}} \rightarrow \boxed{\begin{array}{ll} u = \ln(t) & dv = dt \\ du = \frac{1}{t}dt & v = t \end{array}}$

$\int y\,dy = \frac{1}{2}y^2 = \frac{1}{3}t\ln(t) - \frac{1}{3}t + \frac{C}{3} \Rightarrow y^2 = \frac{2}{3}t\ln(t) - \frac{2}{3}t + \frac{2}{3}C$

$\Rightarrow y^2 = \frac{2}{3}t\ln(t) - \frac{2}{3}t + C$ (✓) ← Again, C arbitrary.

$\downarrow y=2, t=1$

$4 = \frac{2}{3}\ln(1) - \frac{2}{3}1 + C = -\frac{2}{3} + C \Rightarrow C = \frac{14}{3}$

$\Rightarrow y^2 = \frac{2}{3}t\ln(t) - \frac{2}{3}t + \frac{14}{3}$

$\Rightarrow y(t) = \pm\sqrt{\frac{2}{3}t\ln(t) - \frac{2}{3}t + \frac{14}{3}}$

$\Rightarrow \boxed{y(t) = \sqrt{\frac{2}{3}t\ln(t) - \frac{2}{3}t + \frac{14}{3}}}$ since $y(1) = 2 > 0$.

Remark: From step (✓), one can obtain the general solution as:

$$\boxed{y(t) = \pm\sqrt{\frac{2}{3}t\ln(t) - \frac{2}{3}t + C}}$$

Example 3. Write a differential equation describing each of the following types of proportionality:

1. A strain of bacteria grows at a rate (directly) proportional to its population P at time t.

$$\frac{dP}{dt} = kP$$

Change in population

$k>0$ since population grows (k is proportionality constant)

2. A strain of bacteria grows at a rate inversely proportional to its population P at time t.

$$\frac{dP}{dt} = \frac{k}{P}$$

3. The rate at which a group of 8300 people become infected is jointly proportional to the number of people already infected P (at time t) and the people not infected.

$$\frac{dP}{dt} = k\underset{\text{infected}}{P}\underset{\text{not infected.}}{(8300-P)}$$

Example 4. A radioactive element has a half-life of 5 years. If the element initially weighs 4 [redacted], find the amount left after 12 years.

grams

Definition: Half-life is amt. of time it takes for sample to lose ½ its original mass, amt., etc.

Assume $\frac{dP}{dt} = kP$ for these problems $\Rightarrow P(t) = Ce^{kt}$. Given $P(0) = 4, P(5) = 2$, find $P(12)$.

$P(t) = Ce^{kt} \overset{t=0}{\Rightarrow} P(0) = \boxed{C = 4} \Rightarrow P(t) = 4e^{kt} \Rightarrow P(5) = 2 = 4e^{5k} \Rightarrow \frac{1}{2} = e^{5k} \Rightarrow \boxed{k = \frac{\ln(1/2)}{5}}$

$\Rightarrow P(t) = 4e^{\ln(1/2)t/5}$

$\Rightarrow P(12) = 4e^{\ln(1/2)\cdot 12/5} \approx \boxed{0.7578\text{g}}$

Example 5. After 10 minutes in Joe's room, his tea has cooled from 100°C to 50°C. If the room temperature is 20°C, find the temperature 50 minutes later. Round to the nearest hundreth.

Need Newton's Cooling Formula: $\frac{dT}{dt} = k(T-S)$ T = temp. of tea @ time t, S = surrounding temp $= 20$

Solve $\frac{dT}{dt} = k(T-20) \Rightarrow \frac{dT}{T-20} = k\,dt \Rightarrow \ln|T-20| = kt + C$

with $T(0)=100$, $T(10)=50$ to find $T(60)$.

$\downarrow t=0$

$\ln|80| = C \Rightarrow C = \ln(80)$

$\Rightarrow \ln|T-20| = kt + \ln(80)$

$\Rightarrow |T-20| = e^{kt+\ln(80)} = e^{kt}e^{\ln(80)} = 80e^{kt}$

Note $T-20>0$, why? Tea hotter than room, temps can't get colder

$\Rightarrow T-20 = 80e^{kt}$

$\Rightarrow T(t) = 20 + 80e^{kt}$

$\downarrow t=10$

$T(10) = 20 + 80e^{10k} = 50 \Rightarrow e^{10k} = \frac{3}{8}$

$\Rightarrow k = \frac{\ln(3/8)}{10}$

$\Rightarrow \boxed{T(t) = 20 + 80e^{\ln(3/8)t/10}}$

$\Rightarrow \boxed{T(60) \approx 20.22\ ^{\circ}C}$