MATH 16020 Lesson 5: Integration by Parts II

Spring 2021

Recall. Integration by parts formulas:

$\boxed{\int u\,dv = uv - \int v\,du}$

Apply FTC

$\boxed{\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du}$

How to choose u in general?

LIATE!

L - Logarithms
I - Inverse Trig
A - Algebraic
T - Trig
E - Exponentials.

Example 1. Liateé, a fictional coffee shop for mathematicians, has customers getting served at the rate of:

$$P'(t) = 2(t+3)e^{-t/2}$$

where t is in hours since 7:00 AM. How many people get serviced between 9:00AM and 11:00AM? Round to the nearest customer.

$t=2$

$t=4$

$$\Rightarrow \text{Find } \int_2^4 2(t+3)e^{-t/2}\,dt = \left[-4(t+3)e^{-t/2}\right]_2^4 + \int_2^4 +4e^{-t/2}\,dt$$

$\boxed{\begin{array}{ll} u = 2(t+3) & dv = e^{-t/2}\,dt \\ du = 2\,dt & v = -2e^{-t/2} \end{array}}$

$$= \left[-4(t+3)e^{-t/2} - 8e^{-t/2}\right]_2^4$$

$$= \left[-4(t+5)e^{-t/2}\right]_2^4$$

$$= \left(-4(9)e^{-2}\right) - \left(-4(7)e^{-1}\right)$$

$$= -36e^{-2} + 28e^{-1} \approx 5.429 \approx \boxed{5}$$

Example 2. Liateé has a rewards club, and the number of members can be modeled by:

$$N(t) = \frac{6te^{2t} - 4}{e^t} = 6te^{t} - 4e^{-t}$$

where t is the number of months after its opening in January 2018, and $t = 0$ corresponds to January 1st. Find the average number of members during the period consisting of March and April. (Assume that all months have equal duration for this problem.) Round to nearest whole number.

Jan, F | M, A |

$t=0$ 1 2 3 4

$\underbrace{\quad 2 \quad 3 \quad 4 \quad}_{t \text{ in } [2,4]}$

$$\Rightarrow N_{AVG} = \frac{\int_2^4 6te^t - 4e^{-t}\,dt}{4-2}$$

$$= \int_2^4 3te^t - 2e^{-t}\,dt$$

$$= 3\boxed{\int_2^4 te^t\,dt} - 2\boxed{\int_2^4 e^{-t}\,dt}$$

$u = t \quad dv = e^t dt$

$du = dt \quad v = e^t$

$$\left[te^t\right]_2^4 - \int_2^4 e^t\,dt$$

$$= \left[te^t - e^t\right]_2^4$$

$$= \left[4e^4 - e^4\right] - \left[2e^2 - e^2\right]$$

$$= \boxed{3e^4 - e^2}$$

$$\int_2^4 e^{-t}\,dt = \left[-e^{-t}\right]_2^4$$

$$= e^{-2} - e^{-4}$$

$$\Rightarrow N_{AVG} = 3\int_2^4 te^t\,dt - 2\int_2^4 e^{-t}\,dt = 3(3e^4 - e^2) - 2(e^{-2} - e^{-4})$$

$$\approx \boxed{469}$$

Example 3. To attract customers, Liateé bought some antiques to spruce up the shop, and analysts have determined that the probability (from 0 to 1) that an antique has $100x$ percentage of aluminum is modeled by:

$$P(x) = \frac{x}{2\sqrt{2x+3}}$$

where x is also between 0 and 1. Find the probability (from 0 to 1) that an antique has at least 85% aluminum. Round answer to 4 decimal places.

$0 \le x \le 1 \Rightarrow 0 \le 100x \le 100$

As percentages, this makes sense.

Want $100x \ge 85$, or $0.85 \le x$, so with $x \le 1$, we have $\boxed{0.85 \le x \le 1}$

"Add" up all probabilities $P(x)$ for all x in $[0.85, 1]$ via the following integral:

Find $\displaystyle\int_{0.85}^{1} \frac{x}{2\sqrt{2x+3}}\,dx = \int_{0.85}^{1} \frac{x}{2}(2x+3)^{-1/2}\,dx = \left[\frac{x}{2}(2x+3)^{1/2}\right]_{0.85}^{1} - \int_{0.85}^{1} \frac{1}{2}(2x+3)^{1/2}\,dx$

$u = \frac{x}{2}$ $\quad dv = (2x+3)^{-1/2}dx$

$du = \frac{1}{2}dx$ $\quad v = (2x+3)^{1/2}$

$$= \left[\frac{x}{2}(2x+3)^{1/2} - \frac{1}{6}(2x+3)^{3/2}\right]_{0.85}^{1}$$

$$= \left(\frac{1}{2}(5)^{1/2} - \frac{1}{6}(5^{3/2})\right) - \left(\frac{0.85}{2}(2\cdot 0.85+3)^{1/2} - \frac{1}{6}(2\cdot 0.85+3)^{3/2}\right)$$

$\approx \boxed{0.0315}$, or 3.15%

Probability an antique has $\ge 85\%$ aluminum

Remark: Evaluate integrals in this order:

1. Integral from MA16010/lesson R, break up by addition/mult. by a #.
2. If I can use substitution, do it.
3. Else, apply int. by parts
   Last resort!

Example 4. (TIME PERMITTING) Liateé has a rather large menu, and analysts determined that the ability of a customer to memorize this menu is modeled by:

$$M(t) = 13000\frac{\ln(\sqrt{t}+1)}{(t+1)^2}$$

where $13 \leq t \leq 100$ is the customer's age in years and M is on a scale from 1 to 100. Find the average memorization ability of a customer between ages 35 and 48. Round answer to 3 decimal places.

$$M_{AVG} = \frac{\int_{35}^{48} 13000(t+1)^{-2}\ln(\sqrt{t+1})\,dt}{48-35} = \frac{13000}{13}\int_{35}^{48}(t+1)^{-2}\ln(\sqrt{t+1})\,dt$$

$u = \ln(\sqrt{t+1})$ $\quad dv = (t+1)^{-2}\,dt$

$du = \frac{1}{\sqrt{t+1}}\cdot\frac{1}{2\sqrt{t+1}}\,dt$ $\quad v = -(t+1)^{-1}$

$= \frac{1}{2(t+1)}\,dt$

$$= 1000\left[\left[-(t+1)^{-1}\ln(\sqrt{t+1})\right]_{35}^{48} + \int_{35}^{48}\frac{1}{2(t+1)^2}\,dt\right]$$

$$= 1000\left[-\frac{\ln(\sqrt{t+1})}{t+1} - \frac{1}{t+1}\right]_{35}^{48}$$

$$= 1000\left[\left(\frac{-\ln(\sqrt{49})-1}{49}\right) - \left(\frac{-\ln(\sqrt{36})-1}{36}\right)\right]$$

$$\approx \boxed{13.743}$$