MA 16020 Lesson 36: Eigenvalues and eigenvectors II

Recall: The **eigenvector** of a square $(n \times n)$ matrix A is a vector $v \neq 0$ such that:

$$A \cdot v = \lambda \cdot v$$

for some number λ, which is then called an **eigenvalue** of A. To find them, we set up and solve the *characteristic equation*:

$$\det(\lambda \cdot I_n - A) = 0$$

Eigenvalues of A are obtained as solutions of the char. equation. Given an eigenvalue λ, the corresponding eigenvectors are obtained as:

solutions $v \neq 0$ to the matrix equation $(\lambda I_n - A) \cdot v = 0$

Recall: long division of polynomials.

Example: Find all the solutions to the equation

$$x^3 - 7x^2 + 12x - 6 = 0.$$

"guess" $x = 1$: $1^3 - 7 \cdot 1^2 + 12 \cdot 1 - 6 = 0$

Fact : the polynomial $(x-1)$ divides $x^3 - 7x^2 + 12x - 6$

$$x^3 - 7x^2 + 12x - 6 = (x-1) \cdot p(x)$$

$\rightsquigarrow$ need to divide $x^3 - 7x^2 + 12x - 6$ by $(x-1)$

$$\begin{array}{r} x^2 - 6x + 6 \\ (x-1)\,\big)\; x^3 - 7x^2 + 12x - 6 \\ -(x^3 - x^2) \\ \hline 0 \quad -6x^2 + 12x - 6 \\ -(-6x^2 + 6x) \\ \hline 0 \quad 6x - 6 \\ -(6x - 6) \\ \hline 0 \end{array}$$

$\rightsquigarrow p(x) = x^2 - 6x + 6$

$$(x-1)(x^2 - 6x + 6) = 0$$

$\underline{x_1 = 1}$ or $x^2 - 6x + 6 = 0$

$$x_{2,3} = \frac{6 \pm \sqrt{36 - 24}}{2} = \frac{6 \pm \sqrt{12}}{2} = \frac{6 \pm 2\sqrt{3}}{2} = \underline{3 \pm \sqrt{3}}$$

Example: Find the eigenvalues and eigenvectors for the matrix $\begin{bmatrix} -1 & 0 & -2 \\ -12 & -1 & -12 \\ 4 & 0 & 5 \end{bmatrix}$.

1. char equation

$$\begin{vmatrix} \lambda+1 & 0 & 2 \\ 12 & \lambda+1 & 12 \\ -4 & 0 & \lambda-5 \end{vmatrix} = (\lambda+1)^2(\lambda-5)+0+0+8(\lambda+1)-0-0$$

$$= (\lambda^2+2\lambda+1)(\lambda-5)+8\lambda+8 =$$

$$\begin{matrix} \lambda+1 & 0 & 2 \\ 12 & \lambda+1 & 12 \end{matrix}$$

$$= \lambda^3+2\lambda^2+\lambda-5\lambda^2-10\lambda-5+8\lambda+8 =$$

$$= \lambda^3-3\lambda^2-\lambda+3 = 0$$

$\lambda=1$: $1^3-3\cdot 1^2-1+3=0 \rightsquigarrow \lambda_1=1$

divide $\lambda^3-3\lambda^2-\lambda+3$ by $\lambda-1$:

$$\begin{array}{r} \lambda^2-2\lambda-3 \\ (\lambda-1)\overline{)\lambda^3-3\lambda^2-\lambda+3} \\ -(\lambda^3-\lambda^2) \\ \hline 0\ -2\lambda^2-\lambda+3 \\ -(-2\lambda^2+2\lambda) \\ \hline 0\ -3\lambda+3 \\ -(-3\lambda+3) \\ \hline 0 \end{array}$$

$\rightsquigarrow$ need to solve $\lambda^2-2\lambda-3=0$

$$\lambda_{2,3} = \frac{2\pm\sqrt{4+12}}{2} = \frac{2\pm\sqrt{16}}{2} = \frac{2\pm 4}{2}$$

$\lambda_2 = 3$, $\lambda_3 = -1$

eigenvectors

$\lambda = 1$:

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 12 & 2 & 12 & 0 \\ -4 & 0 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 12 & 2 & 12 & 0 \\ -4 & 0 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 12 & 2 & 12 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 12 & 2 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + z = 0 \rightsquigarrow x = -z$

$y = 0$

$z := 1 \rightsquigarrow x = -1$

$y = 0$

$\rightsquigarrow$ eigenvector $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 3$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 12 & 4 & 12 & 0 \\ -4 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 12 & 4 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + \frac{1}{2} z = 0 \rightsquigarrow x = -\frac{1}{2} z$

$y + \frac{3}{2} z = 0 \rightsquigarrow x = -\frac{3}{2} z$

$z := 1 \rightsquigarrow x = -\frac{1}{2}$

$y = -\frac{3}{2}$ $\rightsquigarrow$ eigenvector $\begin{bmatrix} -1/2 \\ -3/2 \\ 1 \end{bmatrix}$

$\lambda = -1$

$$\left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 12 & 0 & 12 & 0 \\ -4 & 0 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 12 & 0 & 12 & 0 \\ -4 & 0 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -4 & 0 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 0 & -6 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x = 0$

$z = 0$

y arbitrary

Choose $y = 1$

$\rightsquigarrow$ eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Example: Find the eigenvalues and eigenvectors for the matrix $\begin{bmatrix} 3 & -1 & 4 \\ 2 & 3 & +8 \\ -1 & 0 & -2 \end{bmatrix}$.

1. char. eqn.

$$\begin{vmatrix} \lambda-3 & 1 & -4 \\ -2 & \lambda-3 & -8 \\ 1 & 0 & \lambda+2 \end{vmatrix} = (\lambda-3)^2(\lambda+2) + 0 - 8 + 4(\lambda-3) - 0 + 2(\lambda+2)$$

$$\begin{matrix} \lambda-3 & 1 & -4 \\ -2 & \lambda-3 & -8 \end{matrix}$$

$$= (\lambda^2-6\lambda+9)(\lambda+2) - 8 + 4\lambda - 12 + 2\lambda + 4$$

$$= \lambda^3 - 6\lambda^2 + 9\lambda + 2\lambda^2 - 12\lambda + 18 - 8 + 4\lambda - 12 + 2\lambda + 4$$

$$= \lambda^3 - 4\lambda^2 + 3\lambda + 2 = 0$$

Try $\lambda = 1$ ... does not work $(1^3 - 4\cdot 1^2 + 3\cdot 1 + 2 = 2)$

$\lambda = -1$ ... does not work $((-1)^3 - 4\cdot(-1)^2 + 3(-1) + 2 = -6)$

$\lambda = 2$ ... works $(2^3 - 4\cdot 2^2 + 3\cdot 2 + 2 = 8 - 16 + 6 + 2 = 0)$

→ divide $\lambda^3 - 4\lambda^2 + 3\lambda + 2$ by $\lambda - 2$:

$$\begin{array}{rl} & \lambda^2 - 2\lambda - 1 \\ (\lambda-2) \,\big) & \lambda^3 - 4\lambda^2 + 3\lambda + 2 \\ & -(\lambda^3 - 2\lambda^2) \\ & 0 - 2\lambda^2 + 3\lambda + 2 \\ & -(-2\lambda^2 + 4\lambda) \\ & 0 - \lambda + 2 \\ & -(-\lambda + 2) \\ & 0 \end{array}$$

~ $\lambda_1 = 2$, ~~and~~ or $\lambda^2 - 2\lambda - 1 = 0$

$$\lambda_{2,3} = \frac{4 \pm \sqrt{4+4}}{2}$$

$$\lambda_{2,3} = \frac{4 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

2. eigenvectors

$\lambda_1 = 2$:

$$\left[\begin{array}{ccc|c} -1 & 1 & -4 & 0 \\ -2 & -1 & -8 & 0 \\ 1 & 0 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ -2 & -1 & -8 & 0 \\ -1 & 1 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 1 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & -4 & 0 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$x + 4z = 0$ ~ $x = -4z$

$y = 0$ $\qquad$ $y = 0$

Set e.g. $z = 1$ → eigenvector $\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow x = -4$

$\lambda =$

2) $\underline{\lambda = 1+\sqrt{2}}$

$$\left[\begin{array}{ccc|c} -2+\sqrt{2} & 1 & -4 & 0\\ -2 & -2+\sqrt{2} & -8 & 0\\ 1 & 0 & 3+\sqrt{2} & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3+\sqrt{2} & 0\\ -2 & -2+\sqrt{2} & -8 & 0\\ -2+\sqrt{2} & 1 & -4 & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3+\sqrt{2} & 0\\ 0 & -2+\sqrt{2} & -2+2\sqrt{2} & 0\\ -2+\sqrt{2} & 1 & -4 & 0\end{array}\right] \xrightarrow{(2-\sqrt{2})R_1+R_3\to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3+\sqrt{2} & 0\\ 0 & -2+\sqrt{2} & -2+2\sqrt{2} & 0\\ 0 & 1 & -\sqrt{2} & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3+\sqrt{2} & 0\\ 0 & 1 & -\sqrt{2} & 0\\ 0 & -2+\sqrt{2} & -2+2\sqrt{2} & 0\end{array}\right] \xrightarrow{(2-\sqrt{2})R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3+\sqrt{2} & 0\\ 0 & 1 & -\sqrt{2} & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$\left((3+\sqrt{2})(2-\sqrt{2}) = 6+2\sqrt{2}-3\sqrt{2}-2 = 4-\sqrt{2},\quad (4-\sqrt{2})+(-4) = -\sqrt{2} \right)$

$(2-\sqrt{2})(-\sqrt{2}) = 2-2\sqrt{2},$

$(2-2\sqrt{2}) + (-2+2\sqrt{2}) = 0$

$\rightsquigarrow x+(3+\sqrt{2})z = 0 \rightsquigarrow x = (-3-\sqrt{2})z$

$y-\sqrt{2}z = 0 \rightsquigarrow y = \sqrt{2}z$

set $z := 1$

$\rightsquigarrow$ eigenvector $\begin{bmatrix} -3-\sqrt{2}\\ \sqrt{2}\\ 1\end{bmatrix}$ //

3) $\underline{\lambda = 1-\sqrt{2}}$

$$\left[\begin{array}{ccc|c} -2-\sqrt{2} & 1 & -4 & 0\\ -2 & -2-\sqrt{2} & -8 & 0\\ 1 & 0 & 3-\sqrt{2} & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3-\sqrt{2} & 0\\ -2 & -2-\sqrt{2} & -8 & 0\\ -2-\sqrt{2} & 1 & -4 & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3-\sqrt{2} & 0\\ 0 & -2-\sqrt{2} & -2-2\sqrt{2} & 0\\ -2-\sqrt{2} & 1 & -4 & 0\end{array}\right] \xrightarrow{(2+\sqrt{2})R_1+R_3\to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3-\sqrt{2} & 0\\ 0 & -2-\sqrt{2} & -2-2\sqrt{2} & 0\\ 0 & 1 & \sqrt{2} & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3-\sqrt{2} & 0\\ 0 & 1 & \sqrt{2} & 0\\ 0 & -2-\sqrt{2} & -2-2\sqrt{2} & 0\end{array}\right] \xrightarrow{(2+\sqrt{2})R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3-\sqrt{2} & 0\\ 0 & 1 & \sqrt{2} & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$\left((3-\sqrt{2})(2+\sqrt{2}) = 6-2\sqrt{2}+3\sqrt{2}-2 = 4+\sqrt{2},\quad 4+\sqrt{2}-4 = \sqrt{2} \right)$

$\rightsquigarrow x+(3-\sqrt{2})z = 0 \rightsquigarrow x = (-3+\sqrt{2})z$

$y+\sqrt{2}z = 0 \rightsquigarrow y = -\sqrt{2}z$

set e.g. $z := 1$ $\rightsquigarrow x = -3+\sqrt{2},\ y = -\sqrt{2}$

$\Longrightarrow$ eigenvector $\begin{bmatrix} -3+\sqrt{2}\\ -\sqrt{2}\\ 1\end{bmatrix}$ //